Lec 21: Fundamental Matrix
Readings

• Szeliski, Chapter 7.2
• “Fundamental matrix song”
Two-view geometry

• Where do epipolar lines come from?

3d point lies somewhere along r

epipolar plane

epipolar line

epipolar line (projection of r)

Image 1

Image 2
Fundamental matrix

• This *epipolar geometry* of two views is described by a Very Special 3x3 matrix $F$, called the *fundamental matrix*

• $F$ maps (homogeneous) *points* in image 1 to *lines* in image 2!

• The epipolar line (in image 2) of point $p$ is: $Fp$

• *Epipolar constraint* on corresponding points: $q^T Fp = 0$
Fundamental matrix

- Two special points: $e_1$ and $e_2$ (the *epipoles*): projection of one camera into the other
- All of the epipolar lines in an image pass through the epipole
Fundamental matrix

- Why does $F$ exist?
- Let’s derive it...
Fundamental matrix

\( K_1 \): intrinsics of camera 1  \( K_2 \): intrinsics of camera 2

\( \tilde{p} = K_1^{-1} p \): ray through \( p \) in camera 1’s (and world) coordinate system

\( \tilde{q} = K_2^{-1} q \): ray through \( q \) in camera 2’s coordinate system

\( R \): rotation of image 2 w.r.t. camera 1
• $\tilde{p}$, $R^T\tilde{q}$, and $t$ are coplanar
• epipolar plane can be represented as $t \times \tilde{p}$

\[(R^T\tilde{q})^T(t \times \tilde{p}) = 0\]
Fundamental matrix

\[(R^T \tilde{q})^T (t \times \tilde{p}) = 0\]

\[\tilde{q}^T R (t \times \tilde{p}) = 0\]
Cross-product as linear operator

**Useful fact:** Cross product with a vector $t$ can be represented as multiplication with a \( (skew-symmetric) \) 3x3 matrix

\[
[t]_\times = \begin{bmatrix}
0 & -t_z & t_y \\
 t_z & 0 & -t_x \\
-t_y & t_x & 0
\end{bmatrix}
\]

\[
t \times \tilde{p} = [t]_\times \tilde{p}
\]
Fundamental matrix

\[ \tilde{q}^T R (t \times \tilde{p}) = 0 \]

\[ \tilde{q}^T R [t] \times \tilde{p} = 0 \]
Fundamental matrix

\[ \tilde{q}^T R [t] \times \tilde{p} = 0 \]

\[ E \tilde{q}^T = 0 \]

the Essential matrix
\[
\tilde{q}^T R [t] \times \tilde{p} = 0
\]

\[
q^T K_2^{-T} R [t] \times K_1^{-1} p = 0
\]

\[
F \leftarrow \text{the Fundamental matrix}
\]
Fundamental matrix

\[ K_1 \text{ : intrinsics of camera 1} \quad K_2 \text{ : intrinsics of camera 2} \]

\[ R \text{ : rotation of image 2 w.r.t. camera 1} \]

\[ q^T K_2^{-T} R [t] \times K_1^{-1} p = 0 \]

\[ F \leftarrow \text{the Fundamental matrix} \]
Fundamental matrix result

$$q^T F p = 0$$

(Longuet-Higgins, 1981)
Properties of the Fundamental Matrix

- $F_p$ is the epipolar line associated with $p$

- $F^T q$ is the epipolar line associated with $q$
Properties of the Fundamental Matrix

• $Fp$ is the epipolar line associated with $p$

• $F^Tq$ is the epipolar line associated with $q$

• $Fe_1 = 0$ and $F^Te_2 = 0$

• All epipolar lines contain epipole
Properties of the Fundamental Matrix

- \( Fp \) is the epipolar line associated with \( p \)

- \( F^T q \) is the epipolar line associated with \( q \)

- \( Fe_1 = 0 \) and \( F^T e_2 = 0 \)

- \( F \) is rank 2
Rectified case

\[
\begin{align*}
\mathbf{R} &= \mathbf{I}_{3 \times 3} \\
\mathbf{t} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \\
\mathbf{E} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\]
Stereo image rectification

- reproject image planes onto a common plane parallel to the line between optical centers
- pixel motion is horizontal after this transformation
- two homographies (3x3 transform), one for each input image reprojection

Original stereo pair

After rectification
Questions?
Alternative Formulation
Homogeneous notation for lines

Recall that a point \((x, y)\) in 2D is represented by the homogeneous 3-vector \(x = (x_1, x_2, x_3)^T\), where \(x = x_1/x_3, y = x_2/x_3\)

A line in 2D is represented by the homogeneous 3-vector

\[
\begin{pmatrix}
  l_1 \\
  l_2 \\
  l_3
\end{pmatrix}
\]

which is the line \(l_1 x + l_2 y + l_3 = 0\).

**Example** represent the line \(y = 1\) as a homogeneous vector.

Write the line as \(-y + 1 = 0\) then \(l_1 = 0, l_2 = -1, l_3 = 1\), and

\(1 = (0, -1, 1)^T\).

Note that \(\mu(l_1 x + l_2 y + l_3) = 0\) represents the same line (only the ratio of the homogeneous line coordinates is significant).

Writing both the point and line in homogeneous coordinates gives

\(l_1 x_1 + l_2 x_2 + l_3 x_3 = 0\)

- **point on line** \(1.x = 0\) or \(1^Tx = 0\) or \(x^T1 = 0\)
• The line \( l \) through the two points \( p \) and \( q \) is \( l = p \times q \)

Proof

\[
\begin{align*}
  l.p &= (p \times q).p = 0 \\
  l.q &= (p \times q).q = 0
\end{align*}
\]

• The intersection of two lines \( l \) and \( m \) is the point \( x = l \times m \)
Algebraic representation of epipolar geometry

We know that the epipolar geometry defines a mapping

\[ x \rightarrow l' \]

- the map only depends on the cameras \( P, P' \) (not on structure)
- it will be shown that the map is linear and can be written as \( l' = Fx \), where \( F \) is a \( 3 \times 3 \) matrix called the fundamental matrix
Derivation of the algebraic expression

Outline

**Step 1**: for a point $x$ in the first image, back-project a ray with camera $P$.

**Step 2**: choose two points on the ray and project into the second image with camera $P'$.

**Step 3**: compute the line through the two image points using the relation $l' = p \times q$.
• choose camera matrices

\[ \mathbf{P} = \mathbf{K} \left[ \mathbf{R} \mid \mathbf{t} \right] \]

- internal calibration
- rotation
- translation

from world to camera coordinate frame

• first camera

\[ \mathbf{P} = \mathbf{K} \left[ \mathbf{I} \mid \mathbf{0} \right] \]

world coordinate frame aligned with first camera

• second camera

\[ \mathbf{P}' = \mathbf{K}' \left[ \mathbf{R} \mid \mathbf{t} \right] \]
Step 1: for a point $x$ in the first image, back project a ray with camera $P = K[I \mid 0]$

A point $x$ back projects to a ray

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = zK^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = zK^{-1}x$$

where $Z$ is the point’s depth, since

$$X(Z) = \begin{pmatrix} zK^{-1}x \\ 1 \end{pmatrix}$$

satisfies

$$PX(Z) = K[I \mid 0]X(Z) = x$$
**Step 2:** choose two points on the ray and project into the second image with camera $P'$

Consider two points on the ray $X(Z) = \begin{pmatrix} zK^{-1}x \\ 1 \end{pmatrix}$

- $Z = 0$ is the camera centre $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $Z = \infty$ is the point at infinity $\begin{pmatrix} K^{-1}x \\ 0 \end{pmatrix}$

Project these two points into the second view

$$P' \begin{pmatrix} 0 \\ 1 \end{pmatrix} = K'[R \mid t] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = K't$$

$$P' \begin{pmatrix} K^{-1}x \\ 0 \end{pmatrix} = K'[R \mid t] \begin{pmatrix} K^{-1}x \\ 0 \end{pmatrix} = K'RK^{-1}x$$
**Step 3:** compute the line through the two image points using the relation \( l' = p \times q \)

Compute the line through the points \( l' = (K't) \times (K'RK^{-1}x) \)

Using the identity \( (Ma) \times (Mb) = M^{-T}(a \times b) \) where \( M^{-T} = (M^{-1})^T = (M^T)^{-1} \)

\[ l' = K'^{-T} \left( t \times (RK^{-1}x) \right) = K'^{-T}[t]_xRK^{-1}x \]

\[ F \]

\[ l' = Fx \quad F = K'^{-T}[t]_xRK^{-1} \]

Points \( x \) and \( x' \) correspond \( (x \leftrightarrow x') \) then \( x'^{T}l' = 0 \)

\[ x'^{T}Fx = 0 \]