

## 2D Geometric Transformations

CS 465 Lecture 7

## Acknowledgement

- Slides: Steve Marschner

## A little quick math background

- Notation for sets, functions, mappings
- Linear transformations
- Matrices
  - Matrix-vector multiplication
  - Matrix-matrix multiplication
- Geometry of curves in 2D
  - Implicit representation
  - Explicit representation

## Implicit representations

- Equation to tell whether we are on the curve  
 $\{\mathbf{v} \mid f(\mathbf{v}) = 0\}$
- Example: line (orthogonal to  $\mathbf{u}$ , distance  $k$  from  $\mathbf{0}$ )  
 $\{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{u} + k = 0\}$
- Example: circle (center  $\mathbf{p}$ , radius  $r$ )  
 $\{\mathbf{v} \mid (\mathbf{v} - \mathbf{p}) \cdot (\mathbf{v} - \mathbf{p}) + r^2 = 0\}$
- Always define boundary of region
  - (if  $f$  is continuous)

## Explicit representations

- Also called *parametric*
- Equation to map domain into plane  
 $\{f(t) \mid t \in D\}$
- Example: line (containing  $\mathbf{p}$ , parallel to  $\mathbf{u}$ )  
 $\{\mathbf{p} + t\mathbf{u} \mid t \in \mathbb{R}\}$
- Example: circle (center  $\mathbf{b}$ , radius  $r$ )  
 $\{\mathbf{p} + r[\cos t \ \sin t]^T \mid t \in [0, 2\pi)\}$
- Like tracing out the path of a particle over time
- Variable  $t$  is the “parameter”

## Transforming geometry

- Move a subset of the plane using a mapping from the plane to itself  
 $S \rightarrow \{T(\mathbf{v}) \mid \mathbf{v} \in S\}$
- Parametric representation:  
 $\{f(t) \mid t \in D\} \rightarrow \{T(f(t)) \mid t \in D\}$
- Implicit representation:  
 $\{\mathbf{v} \mid f(\mathbf{v}) = 0\} \rightarrow \{T(\mathbf{v}) \mid f(\mathbf{v}) = 0\}$   
 $= \{\mathbf{v} \mid f(T^{-1}(\mathbf{v})) = 0\}$

## Translation

- Simplest transformation:  $T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$
- Inverse:  $T^{-1}(\mathbf{v}) = \mathbf{v} - \mathbf{u}$
- Example of transforming circle

## Linear transformations

- One way to define a transformation is by matrix multiplication:  
 $T(\mathbf{v}) = M\mathbf{v}$
- Such transformations are *linear*, which is to say:  
 $T(a\mathbf{u} + \mathbf{v}) = aT(\mathbf{u}) + T(\mathbf{v})$   
(and in fact all linear transformations can be written this way)

## Geometry of 2D linear transformations

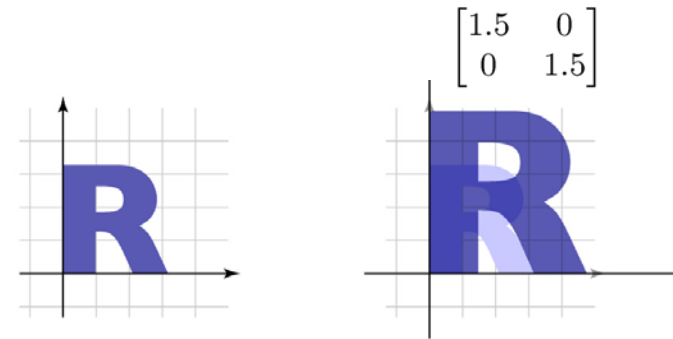
- 2x2 matrices have simple geometric interpretations
  - uniform scale
  - non-uniform scale
  - rotation
  - shear
  - reflection
- Skill: Reading off the matrix

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## Linear transformation gallery

- Uniform scale  $\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} sx \\ sy \end{bmatrix}$



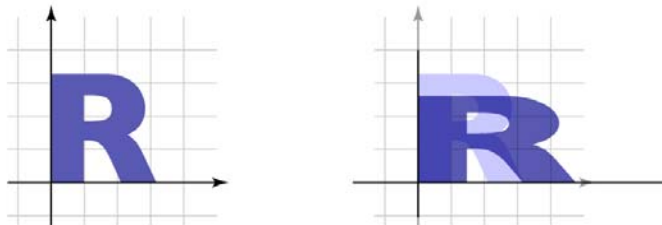
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## Linear transformation gallery

- Nonuniform scale  $\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.8 \end{bmatrix}$$



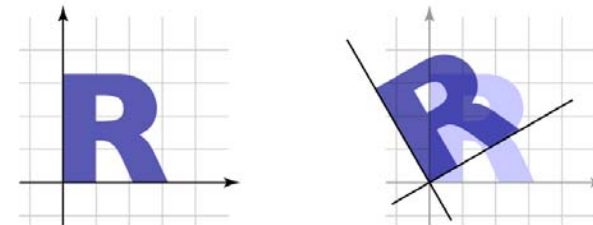
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## Linear transformation gallery

- Rotation  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}$

$$\begin{bmatrix} 0.866 & -.05 \\ 0.5 & 0.866 \end{bmatrix}$$

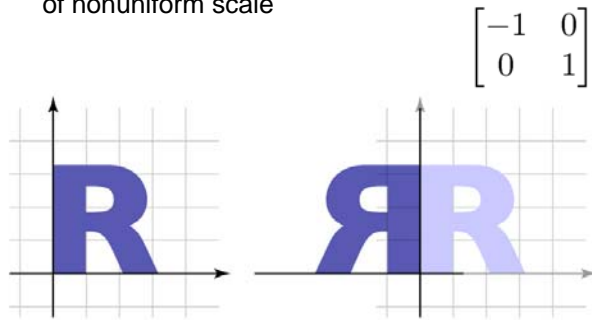


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## Linear transformation gallery

- Reflection
  - can consider it a special case of nonuniform scale

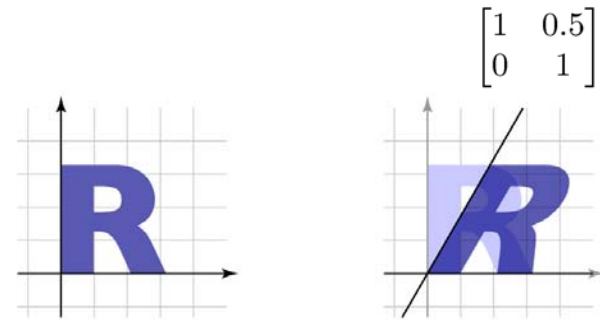


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## Linear transformation gallery

- Shear  $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$



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## Composing transformations

- Want to move an object, then move it some more
 
$$\mathbf{p} \rightarrow T(\mathbf{p}) \rightarrow S(T(\mathbf{p})) = (S \circ T)(\mathbf{p})$$
- We need to represent  $S \circ T$  (“S compose T”)
  - and would like to use the same representation as for S and T
- Translation easy

$$T(\mathbf{p}) = \mathbf{p} + \mathbf{u}_T; S(\mathbf{p}) = \mathbf{p} + \mathbf{u}_S$$

$$(S \circ T)(\mathbf{p}) = \mathbf{p} + (\mathbf{u}_T + \mathbf{u}_S)$$

- Translation by  $\mathbf{u}_T$  then by  $\mathbf{u}_S$  is translation by  $\mathbf{u}_T + \mathbf{u}_S$ 
  - *commutative!* (or, “translation operators commute”)

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## Composing transformations

- Linear transformations also straightforward
  - $T(\mathbf{p}) = M_T \mathbf{p}; S(\mathbf{p}) = M_S \mathbf{p}$
  - $(S \circ T)(\mathbf{p}) = M_S M_T \mathbf{p}$
- Transforming first by  $M_T$  then by  $M_S$  is the same as transforming by  $M_S M_T$ 
  - only *sometimes* commutative
    - e.g., rotations & uniform scales
    - e.g., non-uniform scales w/o rotation
  - Note  $M_S M_T$ , or  $S \circ T$ , is  $T$  first, then  $S$

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## Combining linear with translation

- Need to use both in single framework
- Can represent arbitrary seq. as  $T(\mathbf{p}) = M\mathbf{p} + \mathbf{u}$ 
  - $T(\mathbf{p}) = M_T\mathbf{p} + \mathbf{u}_T$
  - $S(\mathbf{p}) = M_S\mathbf{p} + \mathbf{u}_S$
  - $(S \circ T)(\mathbf{p}) = M_S(M_T\mathbf{p} + \mathbf{u}_T) + \mathbf{u}_S$   
 $= (M_S M_T)\mathbf{p} + (M_S\mathbf{u}_T + \mathbf{u}_S)$
  - e. g.  $S(T(0)) = S(\mathbf{u}_T)$
- Transforming by  $M_T$  and  $\mathbf{u}_T$ , then by  $M_S$  and  $\mathbf{u}_S$ , is the same as transforming by  $M_S M_T$  and  $\mathbf{u}_S + M_S\mathbf{u}_T$ 
  - This will work but is a little awkward

## Homogeneous coordinates

- A trick for getting a more elegant representation
- Extra component  $w$  for vectors, extra row/column for matrices
  - for affine, can always keep  $w = 1$
- Represent linear transformations with dummy extra row and column

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix}$$

## Homogeneous coordinates

- Represent translation using the extra column

$$\begin{bmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t \\ y + s \\ 1 \end{bmatrix}$$

## Homogeneous coordinates

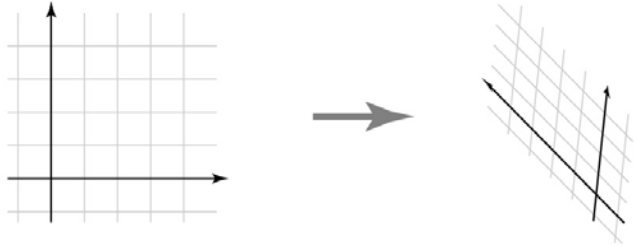
- Composition just works, by 3x3 matrix multiplication

$$\begin{bmatrix} M_S & \mathbf{u}_S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_T & \mathbf{u}_T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} (M_S M_T)\mathbf{p} + (M_S\mathbf{u}_T + \mathbf{u}_S) \\ 1 \end{bmatrix}$$

- Exactly the same as carrying around  $M$  and  $\mathbf{u}$ 
  - but cleaner
  - and generalizes in useful ways as we'll see later

## Affine transformations

- The set of transformations we have been looking at are known as the “*affine*” transformations
  - straight lines preserved; parallel lines preserved
  - ratios of lengths along lines preserved (midpoints preserved)



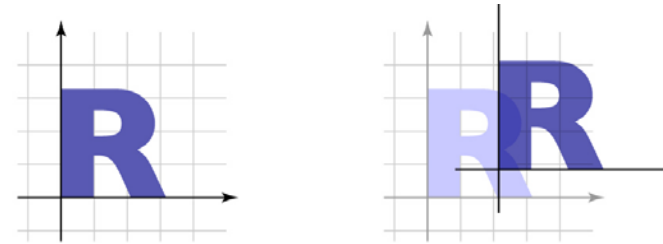
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## Affine transformation gallery

- Translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2.15 \\ 0 & 1 & 0.85 \\ 0 & 0 & 1 \end{bmatrix}$$



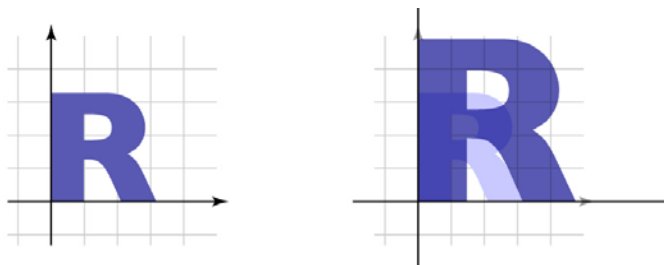
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## Affine transformation gallery

- Uniform scale

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



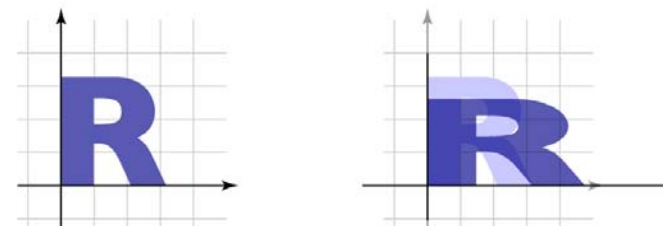
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## Affine transformation gallery

- Nonuniform scale

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

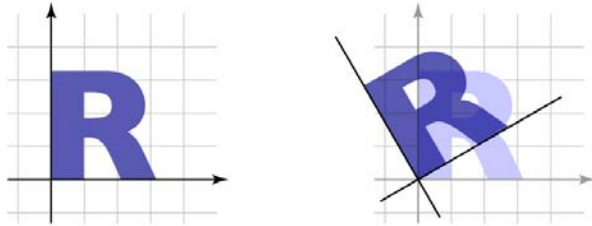


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## Affine transformation gallery

- Rotation  $\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$   $\begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

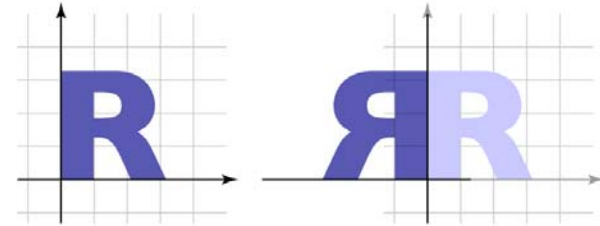


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## Affine transformation gallery

- Reflection  $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$   
 – can consider it a special case of nonuniform scale

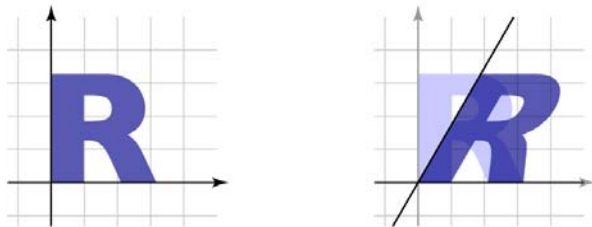


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## Affine transformation gallery

- Shear  $\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$   $\begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



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## General affine transformations

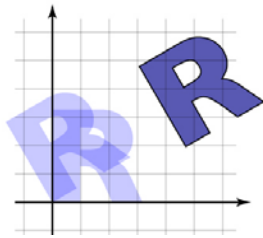
- The previous slides showed “*canonical*” examples of the types of affine transformations
- Generally, transformations contain elements of multiple types
  - often define them as products of canonical transforms
  - sometimes work with their properties more directly

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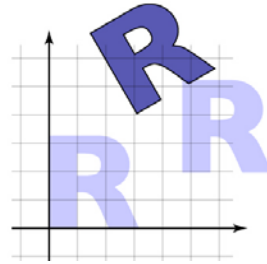
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## Composite affine transformations

- In general **not** commutative: order matters!



rotate, then translate



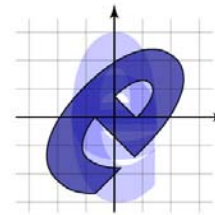
translate, then rotate

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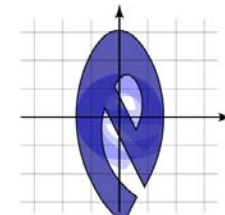
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## Composite affine transformations

- Another example



scale, then rotate



rotate, then scale

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## More math background

- Linear independence and bases
- Orthonormal matrices
- Coordinate systems
  - Expressing vectors with respect to bases
  - Linear transformations as changes of basis

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## Rigid motions

- A transform made up of only translation and rotation is a **rigid motion** or a **rigid body transformation**
- The linear part is an orthonormal matrix

$$R = \begin{bmatrix} Q & \mathbf{u} \\ 0 & 1 \end{bmatrix}$$

- Inverse of orthonormal matrix is transpose
  - so inverse of rigid motion is easy:

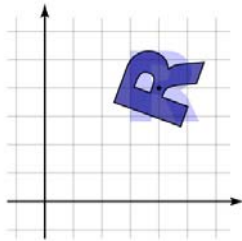
$$R^{-1}R = \begin{bmatrix} Q^T & -Q^T\mathbf{u} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q & \mathbf{u} \\ 0 & 1 \end{bmatrix}$$

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## Composing to change axes

- Want to rotate about a particular point
  - could work out formulas directly...
- Know how to rotate about the origin
  - so translate that point to the origin



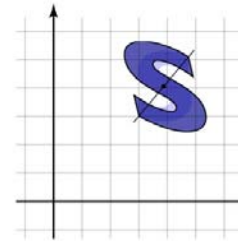
$$M = T^{-1}RT$$

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## Composing to change axes

- Want to scale along a particular axis and point
- Know how to scale along the y axis at the origin
  - so translate to the origin and rotate to align axes



$$M = T^{-1}R^{-1}SRT$$

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## Transforming points and vectors

- Recall distinction points vs. vectors
  - vectors are just offsets (differences between points)
  - points have a location
    - represented by vector offset from a fixed origin
- Points and vectors transform differently
  - points respond to translation, but vectors do not

$$\mathbf{v} = \mathbf{p} - \mathbf{q}$$

$$T(\mathbf{x}) = M\mathbf{x} + \mathbf{t}$$

$$\begin{aligned} T(\mathbf{p} - \mathbf{q}) &= M\mathbf{p} + \mathbf{t} - (M\mathbf{q} + \mathbf{t}) \\ &= M(\mathbf{p} - \mathbf{q}) + (\mathbf{t} - \mathbf{t}) = M\mathbf{v} \end{aligned}$$

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## Transforming points and vectors

- Homogeneous coords. let us exclude translation
  - just put 0 rather than 1 in the last place

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} M\mathbf{p} + \mathbf{t} \\ 1 \end{bmatrix} \quad \begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = \begin{bmatrix} M\mathbf{v} \\ 0 \end{bmatrix}$$

- and note that subtracting two points cancels the extra coordinate, resulting in a vector!

- Preview: projective transformations
  - what's really going on with this last coordinate?
  - think of  $\mathbb{R}^2$  embedded in  $\mathbb{R}^3$ : all affine xfn. preserve  $z=1$  plane
  - could have other transforms; project back to  $z=1$

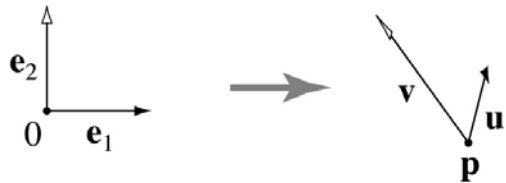
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## Affine change of coordinates

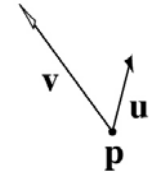
- Six degrees of freedom

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$



## Affine change of coordinates

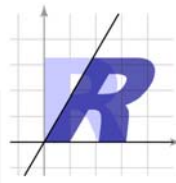
- Coordinate frame: point plus basis
- Interpretation: transformation changes representation of point from one basis to another
- **“Frame to canonical”** matrix has frame in columns
  - takes points represented in frame
  - represents them in canonical basis
  - e.g. [0 0], [1 0], [0 1]
- Seems backward but bears thinking about



$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$

## Affine change of coordinates

- A new way to “read off” the matrix
  - e.g., shear from earlier
  - can look at picture, see effect on basis vectors, write down matrix
- Also an easy way to construct transforms
  - e.g., scale by 2 across direction (1,2)



## Affine change of coordinates

- When we move an object to the origin to apply a transformation, we are really changing coordinates
    - the transformation is easy to express in object’s frame
    - so define it there and transform it
- $$T_e = FT_F F^{-1}$$
- $T_e$  is the transformation expressed w.r.t.  $\{e_1, e_2\}$
  - $T_F$  is the transformation expressed in natural frame
  - $F$  is the frame-to-canonical matrix  $[u \ v \ p]$
  - This is a **similarity transformation**

## Coordinate frame summary

- Frame = point plus basis
- Frame matrix (frame-to-canonical) is

$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$

- Move points to and from frame by multiplying with  $F$

$$p_e = F p_F \quad p_F = F^{-1} p_e$$

- Move transformations using similarity transforms

$$T_e = F T_F F^{-1} \quad T_F = F^{-1} T_e F$$