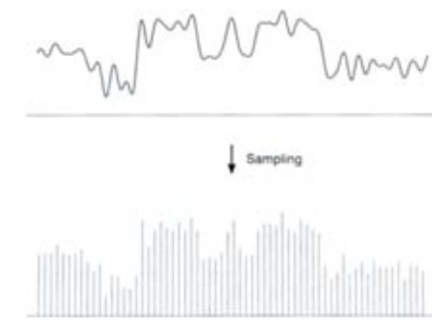
Sampling and reconstruction

CS 465 Lecture 6

Sampled representations

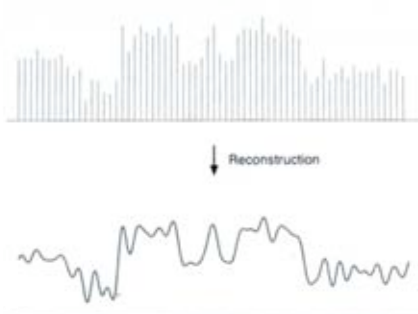
- How to store and compute with continuous functions?
- Common scheme for representation: samples
  - write down the function's values at many points



[FvDFH fig.14.14b / Wolberg]

Reconstruction

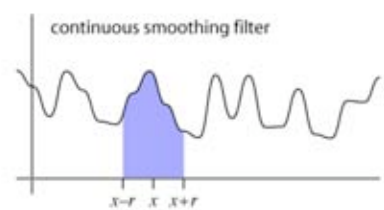
- Making samples back into a continuous function
  - for output (need realizable method)
  - for analysis or processing (need mathematical method)
  - amounts to "guessing" what the function did in between

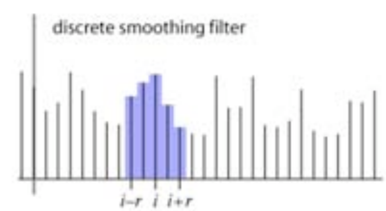


[FvDFH fig.14.14b / Wolberg]

Filtering

- Processing done on a function
  - can be executed in continuous form (e.g. analog circuit)
  - but can also be executed using sampled representation
- Simple example: smoothing by averaging



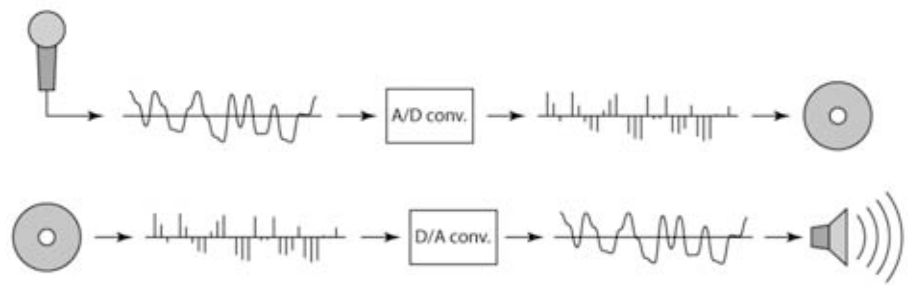


Roots of sampling

- Nyquist 1928; Shannon 1949
  - famous results in information theory
- 1940s: first practical uses in telecommunications
- 1960s: first digital audio systems
- 1970s: commercialization of digital audio
- 1982: introduction of the Compact Disc
  - the first high-profile consumer application
- This is why all the terminology has a communications or audio "flavor"
  - early applications are 1D; for us 2D (images) is important

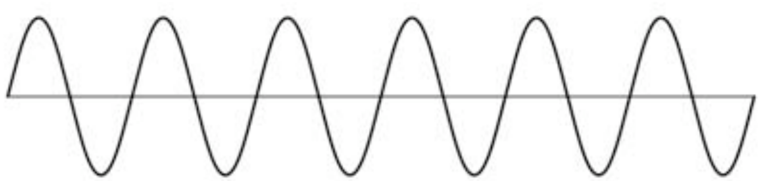
Sampling in digital audio

- Recording: sound to analog to samples to disc
- Playback: disc to samples to analog to sound again
  - how can we be sure we are filling in the gaps correctly?



Undersampling

- What if we "missed" things between the samples?
- Simple example: undersampling a sine wave
  - unsurprising result: information is lost
  - surprising result: indistinguishable from lower frequency
  - also was always indistinguishable from higher frequencies
  - *aliasing*: signals "traveling in disguise" as other frequencies



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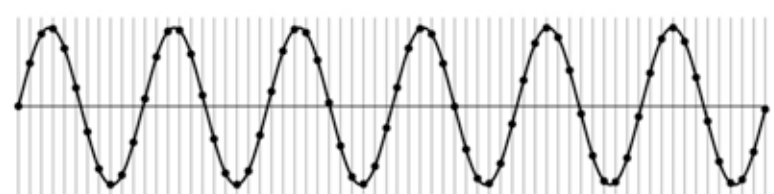

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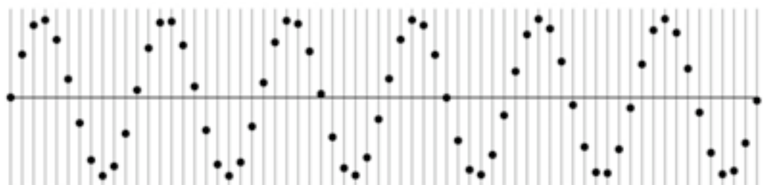

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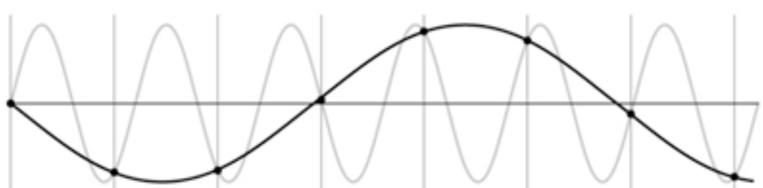

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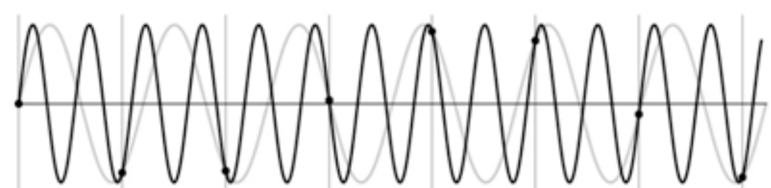
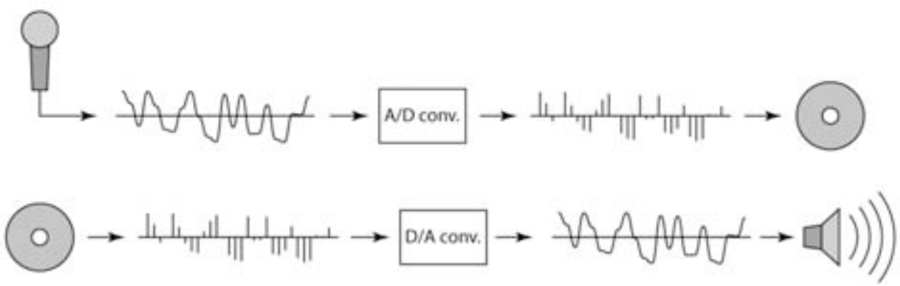


Preventing aliasing

- Introduce lowpass filters:
  - remove high frequencies leaving only safe, low frequencies
  - choose lowest frequency in reconstruction (disambiguate)



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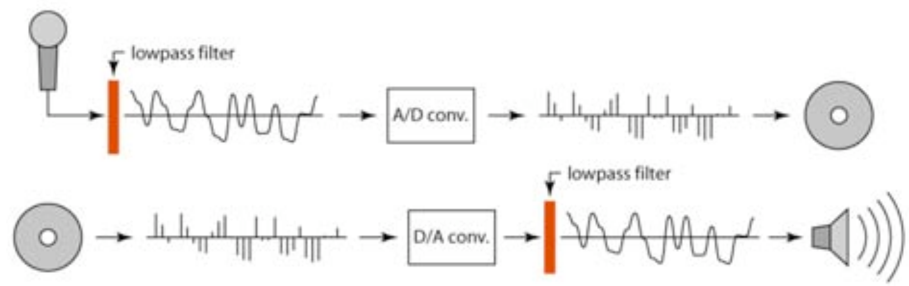
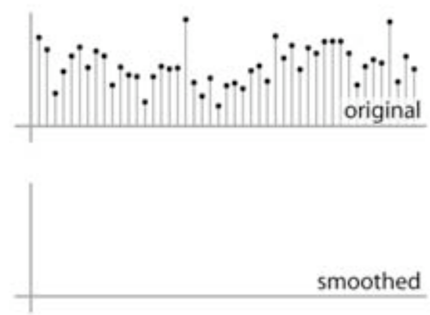


Linear filtering: a key idea

- Transformations on signals; e.g.:
  - bass/treble controls on stereo
  - blurring/sharpening operations in image editing
  - smoothing/noise reduction in tracking
- Key properties
  - linearity: filter($f + g$) = filter(f) + filter(g)
  - shift invariance: behavior invariant to shifting the input
    - delaying an audio signal
    - sliding an image around
- Can be modeled mathematically by *convolution*

Convolution warm-up

- basic idea: define a new function by averaging over a sliding window
- a simple example to start off: smoothing



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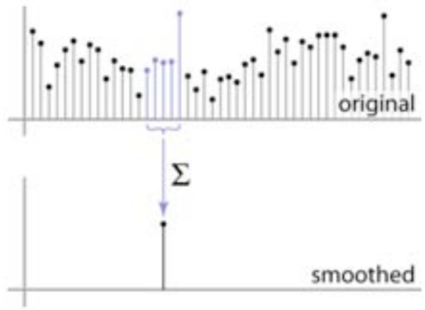

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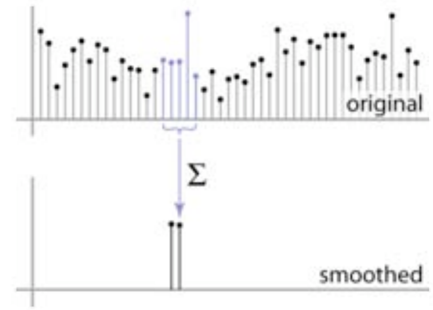

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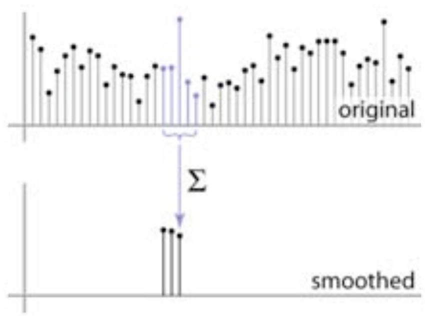

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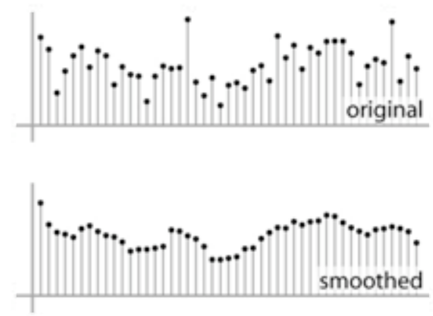

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Convolution warm-up

- Same moving average operation, expressed mathematically:

$$b_{\text{smooth}}[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} b[j]$$

Discrete convolution

- Simple averaging:

$$b_{\text{smooth}}[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} b[j]$$

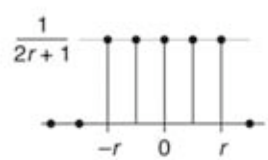
  - every sample gets the same weight
- Convolution: same idea but with *weighted* average

$$(a \star b)[i] = \sum_j a[j]b[i-j]$$

  - each sample gets its own weight (normally zero far away)
- This is all convolution is: it is a **moving weighted average**

Filters

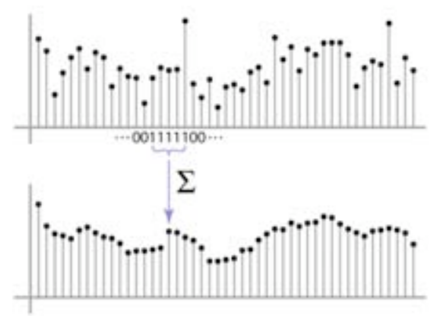
- Sequence of weights *a*[*j*] is called a *filter*
- Filter is nonzero over its *region of support*
  - usually centered on zero: support radius *r*
- Filter is *normalized* so that it sums to 1.0
  - this makes for a weighted average, not just any old weighted sum
- Most filters are symmetric about 0
  - since for images we usually want to treat left and right the same



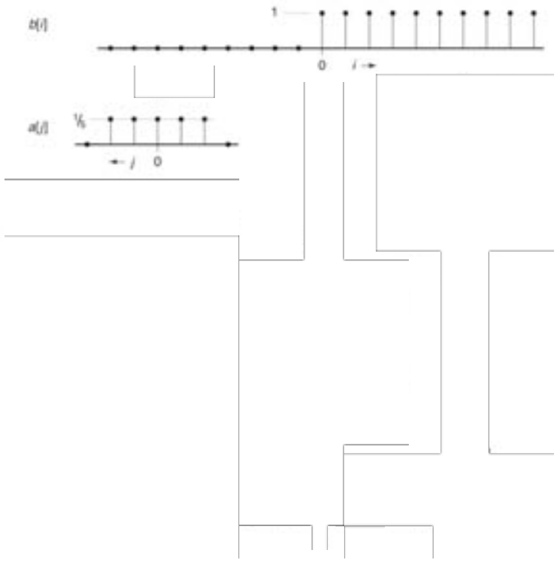
a box filter

Convolution and filtering

- Can express sliding average as convolution with a *box filter*
- a_{box} = […, 0, 1, 1, 1, 1, 1, 0, …]



Example: box and step



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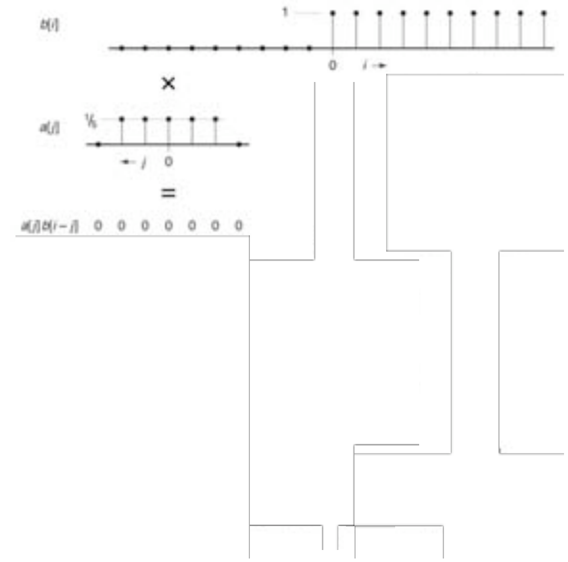

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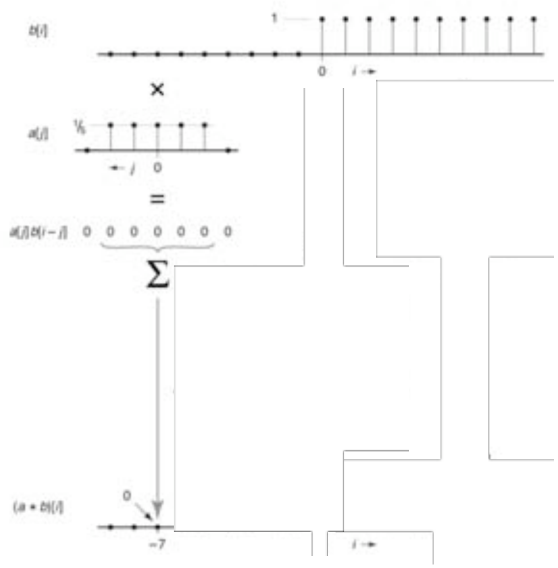

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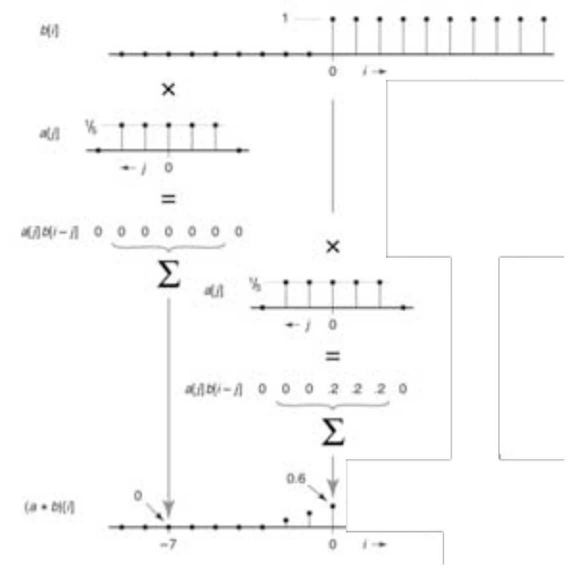

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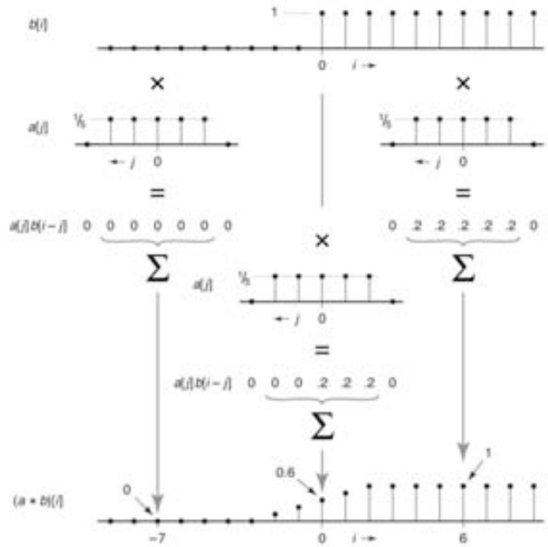
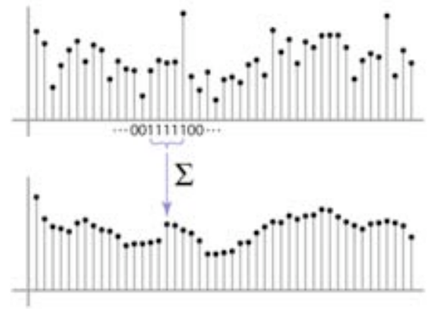


Convolution and filtering

- Convolution applies with any sequence of weights
- Example: bell curve (gaussian-like) [..., 1, 4, 6, 4, 1, ...]



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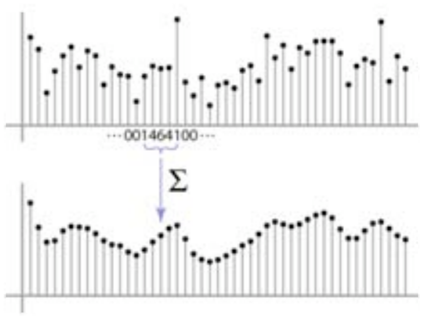

And in pseudocode...

function convolve(sequence a, sequence b, int r, int i)
  $s = 0$
  for $j = -r$ to r
    $s = s + a[j]b[i - j]$
  return s

Discrete convolution

- Notation: $b = c \star a$
- Convolution is a multiplication-like operation
  - commutative $a \star b = b \star a$
  - associative $a \star (b \star c) = (a \star b) \star c$
  - distributes over addition $a \star (b + c) = a \star b + a \star c$
  - scalars factor out $\alpha a \star b = a \star \alpha b = \alpha(a \star b)$
  - identity: unit impulse e = [..., 0, 0, 1, 0, 0, ...]

    $a \star e = a$
- Conceptually no distinction between filter and signal

Discrete filtering in 2D

- Same equation, one more index

$$(a \star b)[i, j] = \sum_{i', j'} a[i', j'] b[i - i', j - j']$$

  - now the filter is a rectangle you slide around over a grid of numbers
- Commonly applied to images
  - blurring (using box, using gaussian, ...)
  - sharpening (impulse minus blur)
- Usefulness of associativity
  - often apply several filters one after another: $(((a * b_1) * b_2) * b_3)$
  - this is equivalent to applying one filter: $a * (b_1 * b_2 * b_3)$

And in pseudocode...

function convolve2d(filter2d a, filter2d b, int i, int j)

$s = 0$

$r = a$.radius

for $i' = -r$ to r **do**

  for $j' = -r$ to r **do**

    $s = s + a[i'][j']b[i - i'][j - j']$

return s

Optimization: separable filters

- basic alg. is $O(r^2)$: large filters get expensive fast!
- definition: $a_2(x,y)$ is *separable* if it can be written as:

$$a_2[i,j] = a_1[i]a_1[j]$$

  - this is a useful property for filters because it allows factoring:

$$\begin{aligned}(a_2 \star b)[i,j] &= \sum_{i'}\sum_{j'} a_2[i',j']b[i-i',j-j'] \\ &= \sum_{i'}\sum_{j'} a_1[i']a_1[j']b[i-i',j-j'] \\ &= \sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j']b[i-i',j-j'] \right)\end{aligned}$$

Separable filtering

$$a_2[i,j] = a_1[i]a_1[j]$$

1	4	6	4	1
4	16	24	16	4
6	24	36	24	6
4	16	24	16	4
1	4	6	4	1

0	0	0	0	0
0	0	0	0	0
1	4	6	4	1
0	0	0	0	0
0	0	0	0	0

0	0	1	0	0
0	0	4	0	0
0	0	6	0	0
0	0	4	0	0
0	0	1	0	0

first, convolve with this

$$\sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j']b[i-i',j-j'] \right)$$

Separable filtering

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0	0	0	0	0

0	0	1	0	0
0	0	4	0	0
0	0	6	0	0
0	0	4	0	0
0	0	1	0	0

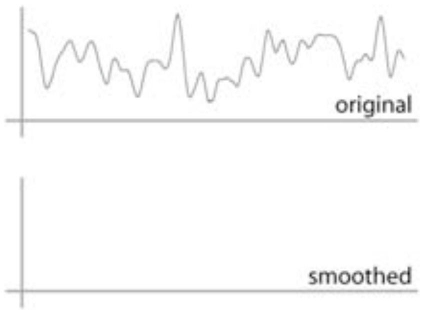
second, convolve with this

first, convolve with this

$$\sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j']b[i-i',j-j'] \right)$$

Continuous convolution: warm-up

- Can apply sliding-window average to a continuous function just as well
  - output is continuous
  - integration replaces summation



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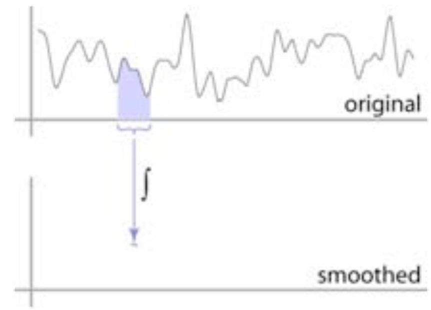

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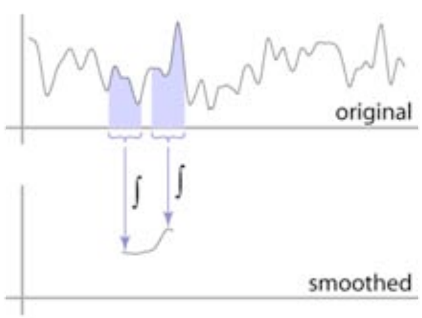

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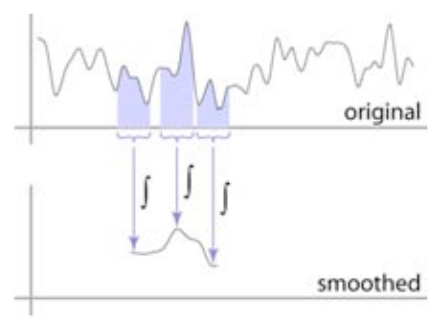

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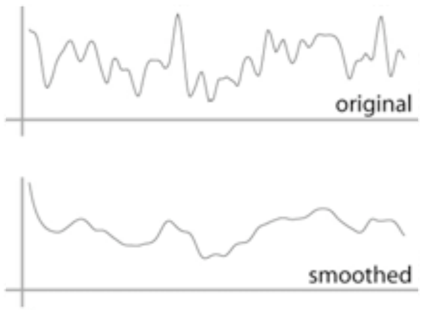


Continuous convolution

- Sliding average expressed mathematically:

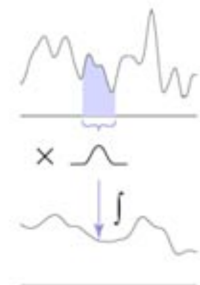
$$g_{\text{smooth}}(x) = \frac{1}{2r}\int_{x-r}^{x+r} g(t)dt$$

  - note difference in normalization (only for box)

- Convolution just adds weights

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

  - weighting is now by a function
  - weighted integral is like weighted average
  - again bounds are set by support of *f(x)*



One more convolution

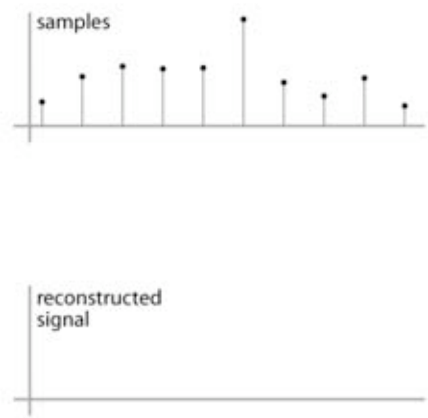
- Continuous–discrete convolution

$$(a \star f)(x) = \sum_i a[i]f(x-i)$$

$$(a \star f)(x,y) = \sum_{i,j} a[i,j]f(x-i,y-j)$$

  - used for reconstruction and resampling

Continuous-discrete convolution



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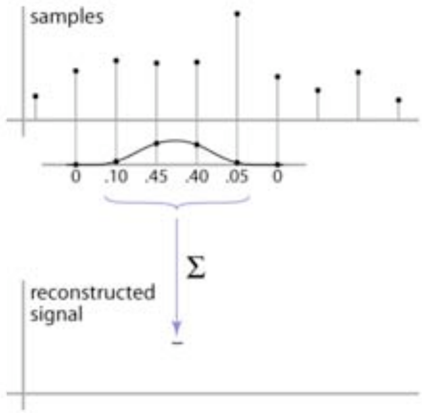

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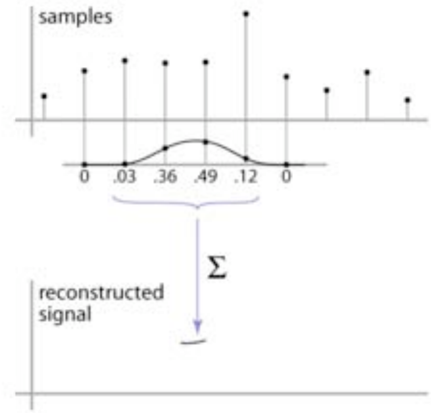

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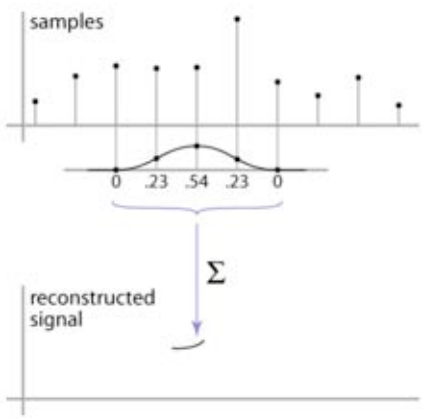

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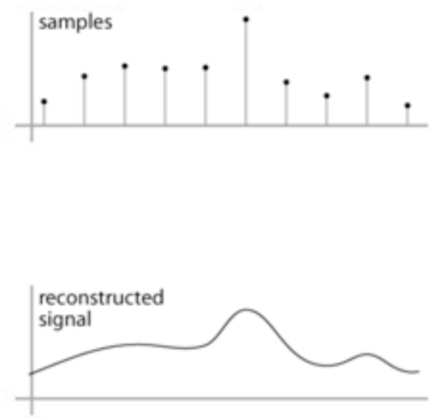
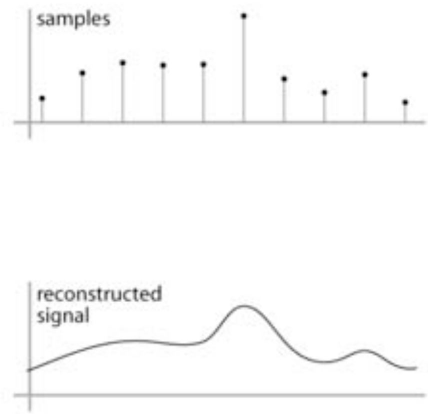


Resampling

- Changing the sample rate
  - in images, this is enlarging and reducing
- Creating more samples:
  - increasing the sample rate
  - "upsampling"
  - "enlarging"
- Ending up with fewer samples:
  - decreasing the sample rate
  - "downsampling"
  - "reducing"

Resampling

- Reconstruction creates a continuous function
  - forget its origins, go ahead and sample it



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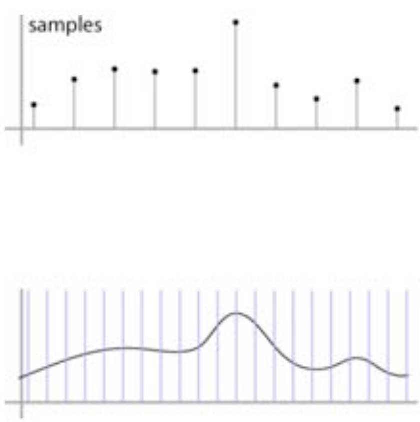

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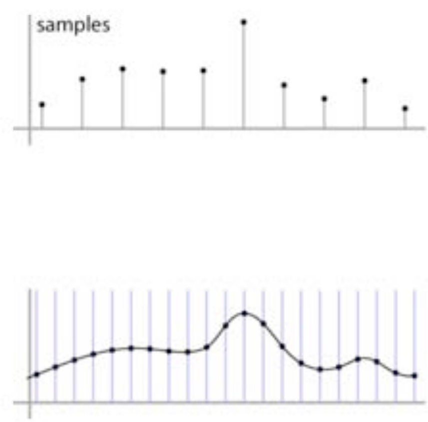

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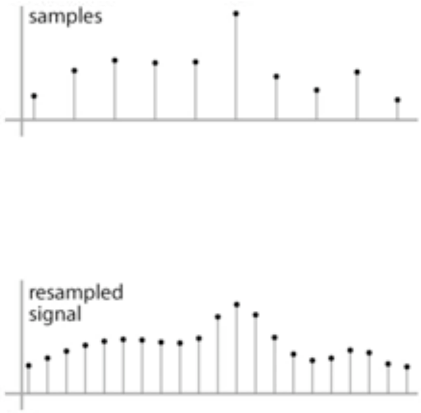

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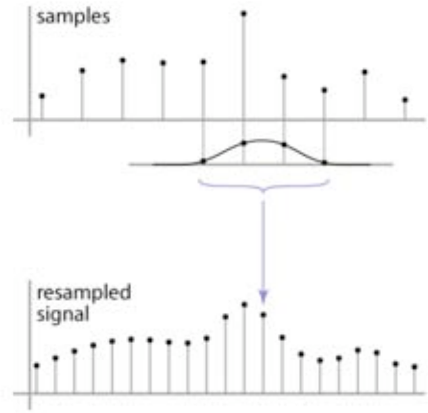


And in pseudocode...

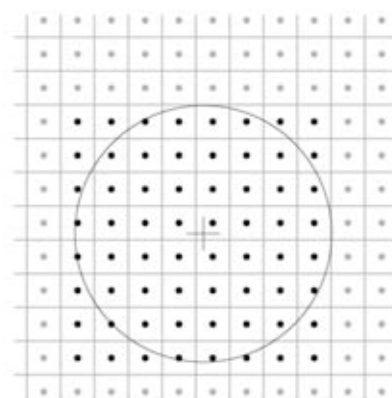
function reconstruct(sequence a, filter f, real x)
$s = 0$
$r = f.\text{radius}$
for $i = \lceil x - r \rceil$ to $\lfloor x + r \rfloor$ **do**
  $s = s + a[i]f(x - i)$
return s

Cont.–disc. convolution in 2D

- same convolution—just two variables now

$$(a \star f)(x, y) = \sum_{i,j} a[i, j] f(x - i, y - j)$$

  - loop over nearby pixels, average using filter weight
  - looks like discrete filter, but offsets are not integers and filter is continuous
  - remember placement of filter relative to grid is variable



Cont.–disc. convolution in 2D

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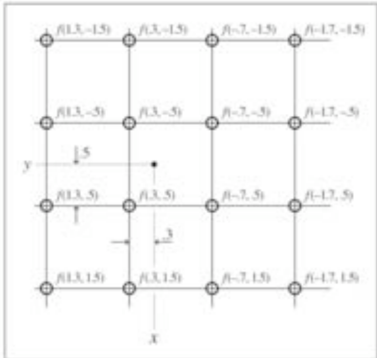
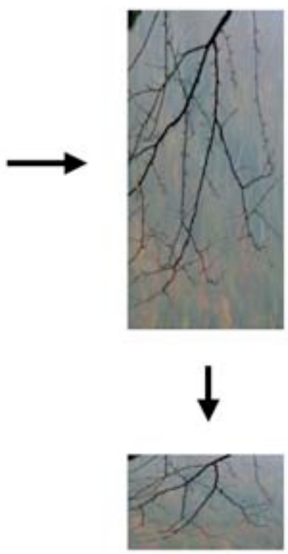


Separable filters for resampling

- just as in filtering, separable filters are useful
  - separability in this context is a statement about a continuous filter, rather than a discrete one:

    $$f_2(x, y) = f_1(x) f_1(y)$$
- resample in two passes, one resampling each row and one resampling each column
- intermediate storage required: product of one dimension of src. and the other dimension of dest.
- same yucky details about boundary conditions



two-stage resampling using a separable filter

A gallery of filters

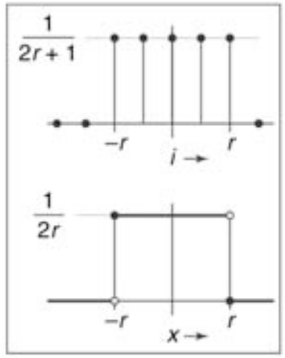
- Box filter
  - Simple and cheap
- Tent filter
  - Linear interpolation
- Gaussian filter
  - Very smooth antialiasing filter
- B-spline cubic
  - Very smooth
- Catmull-rom cubic
  - interpolating

Mitchell-Netravali cubic
Good for image upsampling

Box filter

$$a_{\mathrm{box},r}[i] = \begin{cases} 1/(2r+1) & |i| \le r, \\ 0 & \text{otherwise.} \end{cases}$$

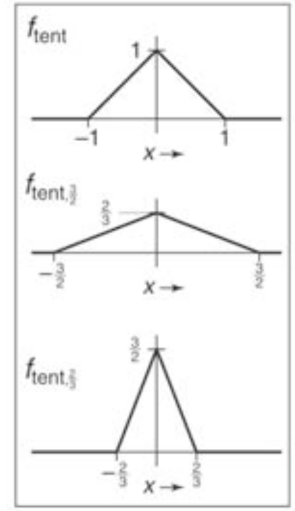
$$f_{\mathrm{box},r}(x) = \begin{cases} 1/(2r) & -r \le x < r, \\ 0 & \text{otherwise.} \end{cases}$$



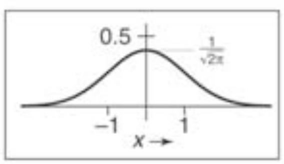
Tent filter

$$f_{\mathrm{tent}}(x) = \begin{cases} 1 - |x| & |x| < 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$f_{\mathrm{tent},r}(x) = \frac{f_{\mathrm{tent}}(x/r)}{r}.$$

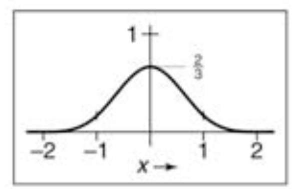


Gaussian filter



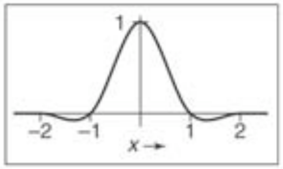
$$f_g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

B-Spline cubic



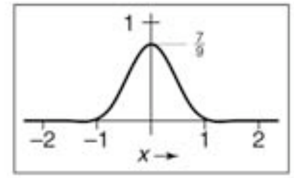
$$f_B(x) = \frac{1}{6} \begin{cases} -3(1-|t|)^3 + 3(1-|t|)^2 + 3(1-|t|) + 1 & -1 \le t \le 1, \\ (2-|t|)^3 & 1 \le |t| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Catmull-Rom cubic



$$f_C(x) = \frac{1}{2}\begin{cases} -3(1-|t|)^3 + 4(1-|t|)^2 + (1-|t|) & -1 \le t \le 1, \\ (2-|t|)^3 - (2-|t|)^2 & 1 \le |t| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Michell-Netravali cubic



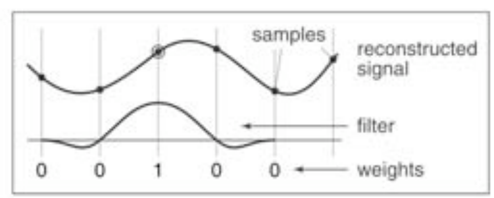
$$\begin{aligned} f_M(x) &= \frac{1}{3} f_B(x) + \frac{2}{3} f_C(x) \\ &= \frac{1}{18}\begin{cases} -21(1-|x|)^3 + 27(1-|x|)^2 + 9(1-|x|) + 1 & -1 \le x \le 1 \\ 7(2-|x|)^3 - 6(2-|x|)^2 & 1 \le |x| \le 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Effects of reconstruction filters

- For some filters, the reconstruction process winds up implementing a simple algorithm
- Box filter (radius 0.5): nearest neighbor sampling
  - box always catches exactly one input point
  - it is the input point nearest the output point
  - so output[*i*, *j*] = input[round(*x*(*i*)), round(*y*(*j*))]
    x(*i*) computes the position of the output coordinate *i* on the input grid
- Tent filter (radius 1): linear interpolation
  - tent catches exactly 2 input points
  - weights are *a* and (1 − *a*)
  - result is straight-line interpolation from one point to the next

Properties of filters

- Degree of continuity
- Impulse response
- Interpolating or no
- Ringing, or overshoot

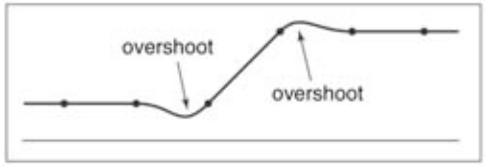


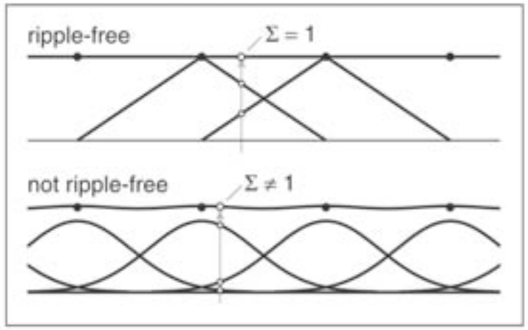
interpolating filter used for reconstruction

Ringing, overshoot, ripples

- Overshoot
  - caused by negative filter values
- Ripples
  - constant in, non-const. out
  - ripple free when:

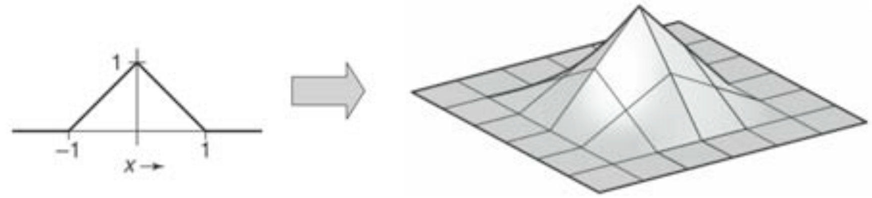
$$\sum_i f(x+i) = 1 \quad \text{for all } x.$$

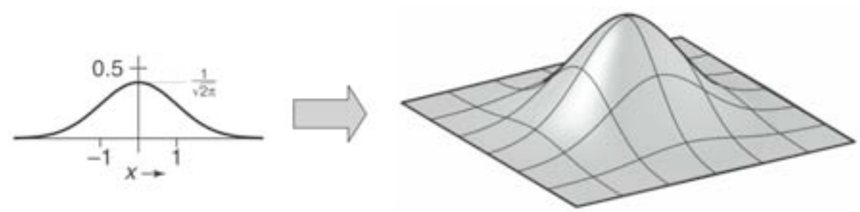




Constructing 2D filters

- Separable filters (most common approach)





Yucky details

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - clip filter (black)
    - wrap around
    - copy edge
    - reflect across edge
    - vary filter near edge

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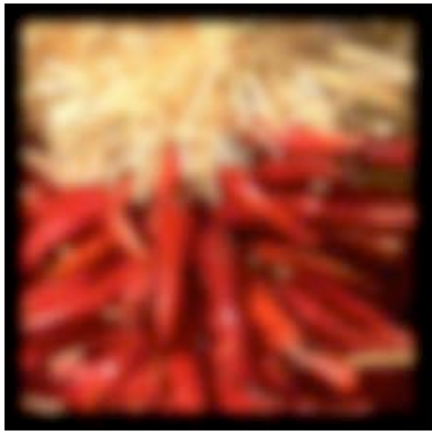

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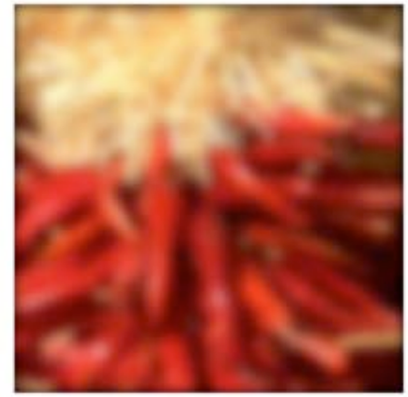

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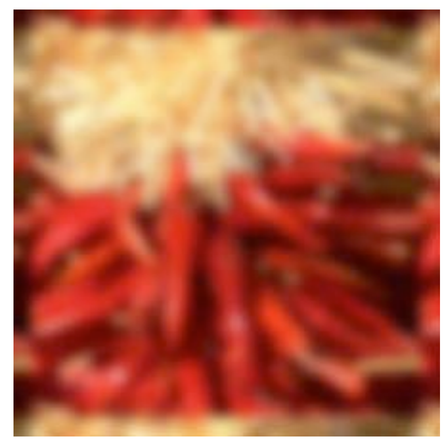

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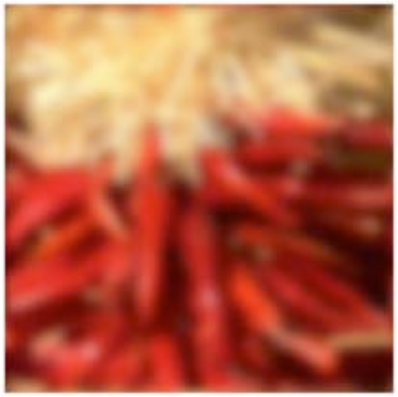

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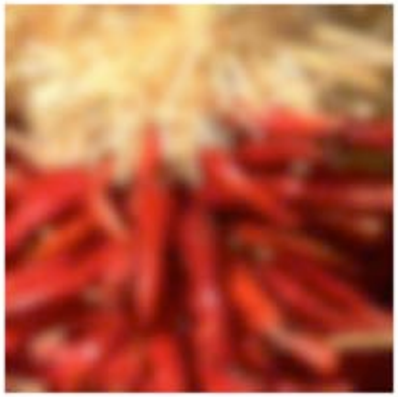
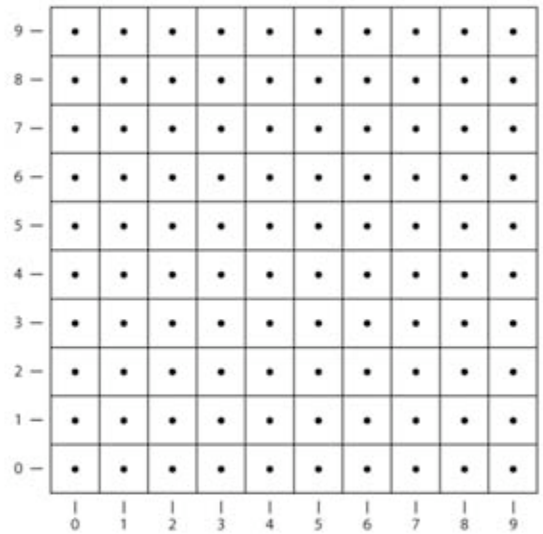


Reducing and enlarging

- Very common operation
  - devices have differing resolutions
  - applications have different memory/quality tradeoffs
- Also very commonly done poorly
- Simple approach: drop/replicate pixels
- Correct approach: use resampling

Resampling example



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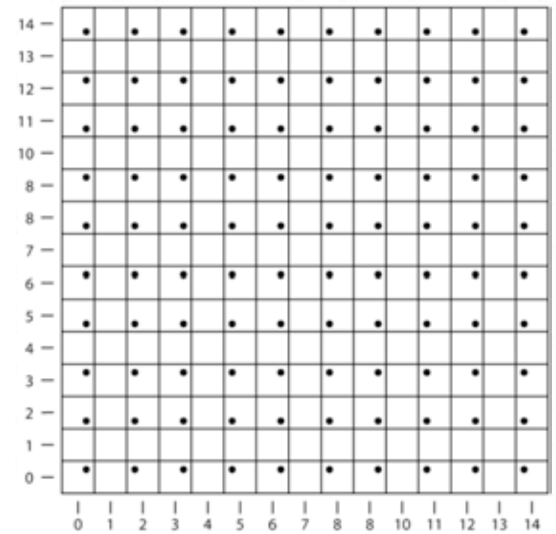

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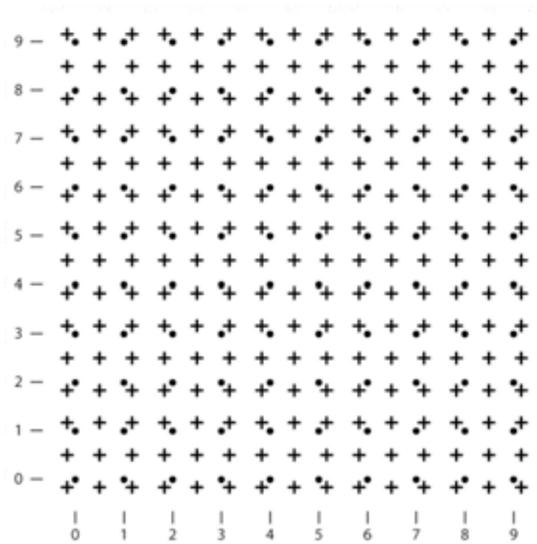

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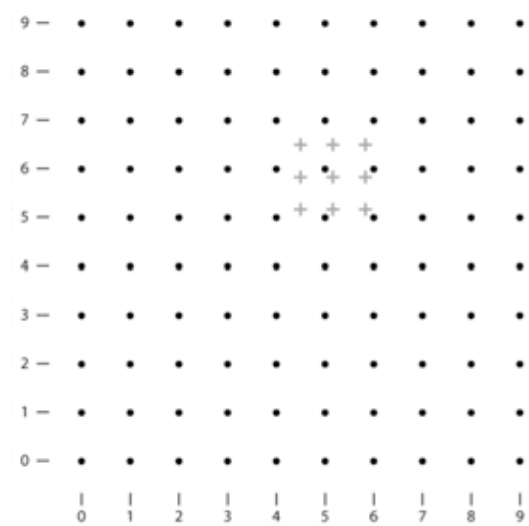

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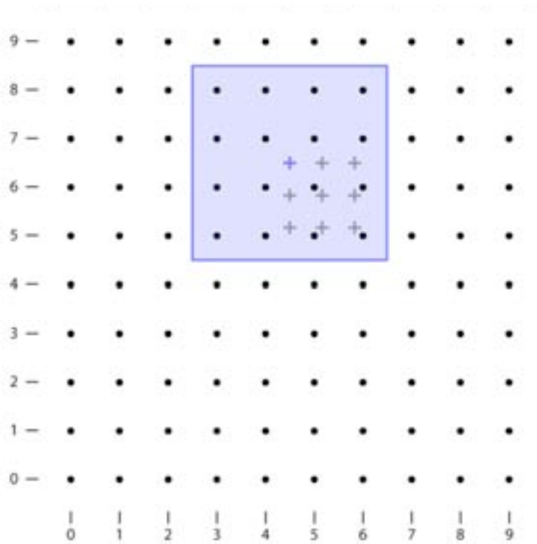

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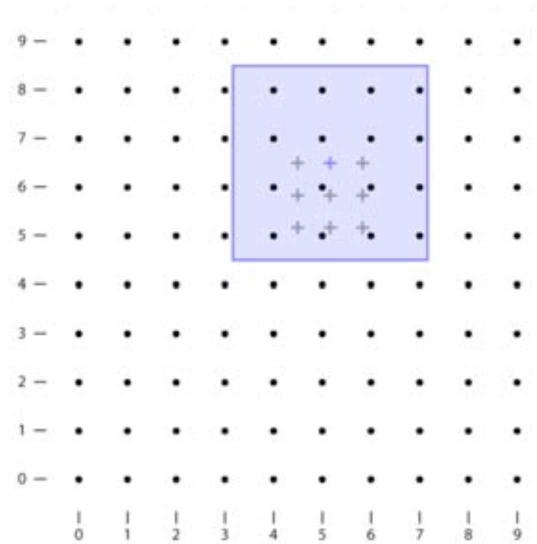

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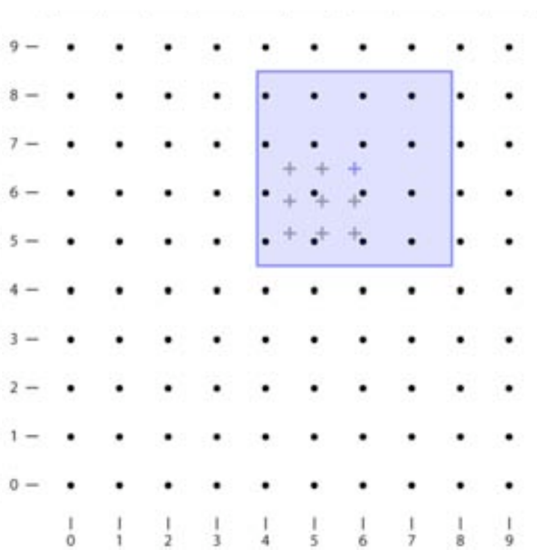

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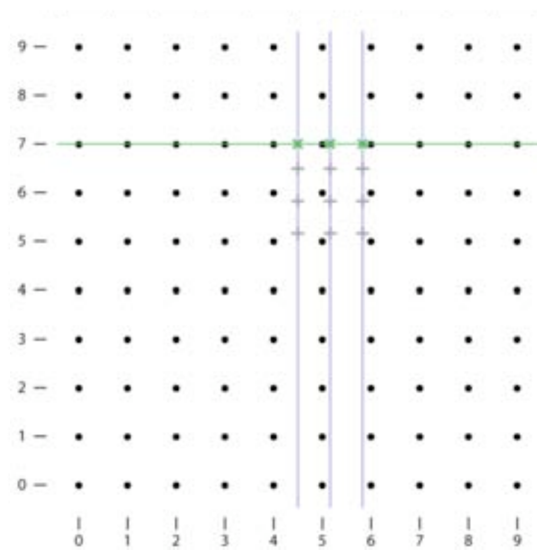

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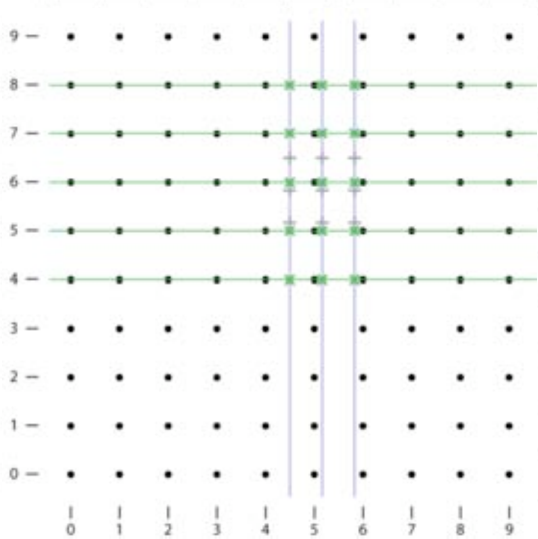

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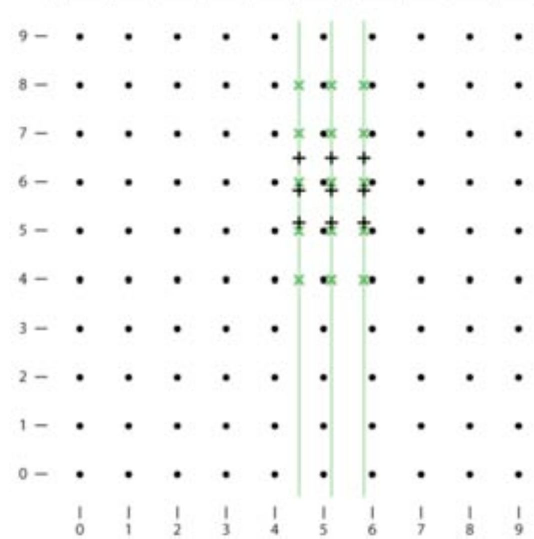

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[Philip Greenspun]

1000 pixel width

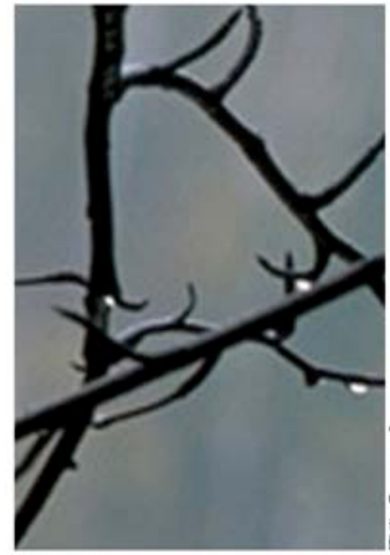
[Philip Greenspun]

by dropping pixels

gaussian filter

250 pixel width

box reconstruction filter



bicubic reconstruction filter

[Philip Greenspun]

4000 pixel width

Types of artifacts

- Garden variety
  - what we saw in this natural image
  - fine features become jagged or sparkle
- Moiré patterns

[Hearn & Baker cover]

600ppi scan of a color halftone image

by dropping pixels　　　gaussian filter

downsampling a high resolution scan

Types of artifacts

- Garden variety
  - what we saw in this natural image
  - fine features become jagged or sparkle
- Moiré patterns
  - caused by repetitive patterns in input
  - produce large-scale artifacts; highly visible
- These artifacts are *aliasing* just like in the audio example earlier
- How do I know what filter is best at preventing aliasing?
  - practical answer: experience
  - theoretical answer: there is another layer of cool math behind all this
    - based on Fourier transforms
    - provides much insight into aliasing, filtering, sampling, and reconstruction
  - learn about it in a signal processing class or in CS 467