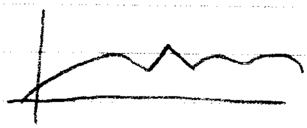
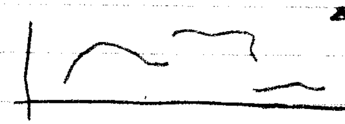
Curve & Surface basics.

First, continuity. A function $\mathbb{R}^n \to \mathbb{R}^m$ is continuous if it has no sudden jumps in its value.

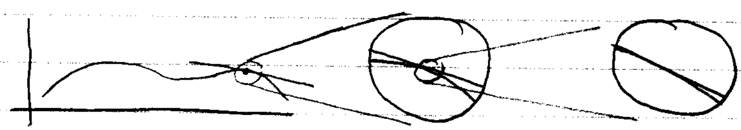
e.g 

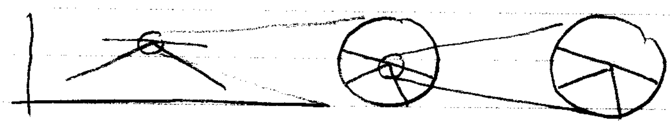
not e.g 

piecewise continuous though

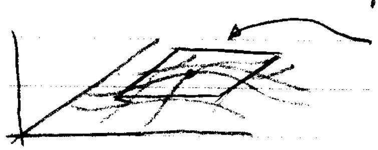
The set of continuous functions is called C^0.

A continuous function is smooth if it can be locally approximated by a straight line (or plane, etc. for $n \geq 1$) at any point in the domain. "Approximated" means that if you zoom in enough the curve becomes indistinguishable from the line.

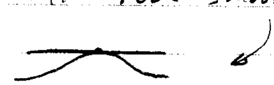
e.g  ...

not e.g.  ...

In a function $\mathbb{R}^2 \to \mathbb{R}$ the approximation is with a plane.



plane is tangent @ this pt.
any slice will look smooth also.



The slope of the line is the derivative. (hence term differentiable)
Eg. if $f(x)$ approx. by $ax+b$ then a is the derivative (f')
if $f(x_1, x_2)$ approx. by $a_1 x_1 + a_2 x_2 + b = \underline{a} \cdot \underline{x} + b$ then $\underline{a} = (a_1, a_2)$ is the derivative. The derivative of a function $f: \mathbb{R}^n \to \mathbb{R}$ is denoted ∇f, and is an n-vector. (called the "gradient")

A function with a continuous derivative is in the class $C^1 \subset C^0$.
(basically this just means it has a derivative everywhere).

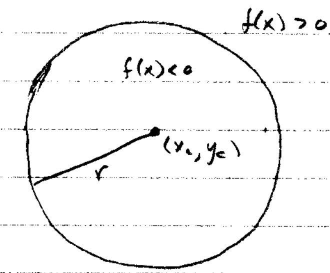
If the derivative is C^k then the function is C^{k+1}. So if the first and second derivatives are continuous then the fn. is C^2.

Curves in 2D.

Implicit curves are specified by $f: \mathbb{R}^2 \to \mathbb{R}$ and are the set of points where $f(x,y)=0$. i.e. $\{\underline{x} \in \mathbb{R}^2 \mid f(\underline{x})=0\}$ – preimage of $\{0\}$

e.g. $f(x-x_c, y-y_c) = \underbrace{(x-x_c)^2 + (y-y_c)^2}_{\text{Distance}^2\ (x,y) \text{ to } (x_c,y_c)} - r^2 = 0$

This is a circle: ⟶



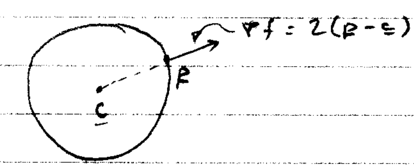
cleaner way to write: $p=(x,y)$, $c=(c_x,c_y)$

$(p-c)\cdot(p-c) - r^2 = 0$

$\|p-c\|^2 - r^2 = 0$ ← easier to see it's a circle.

- Note f has values everywhere, not just on the curve.
  Analogy: topography — elevation is f; sea level is zero. shoreline is the curve $f(\underline{x})=0$.
- Note that αf represents the same curve as f. ($\alpha \neq 0$)
- Nice property: easy to tell whether a point is inside ($f(\underline{x})<0$) or outside ($f(\underline{x})>0$).

Derivative (gradient) of f gives the normal to the implicit curve. (e.g. in circle $\nabla f = (2(x-x_c), 2(y-y_c))$, a vector pointing from $\underline{c}$ to $\underline{p}$:



Why? We know the topography is locally approx'd by a plane, so zoom in sufficiently. ∇f gives the direction of steepest ascent; in fact, the slope in direction $\underline{d}$ is $(\nabla f \cdot \underline{d})$. So when that is zero, when $\nabla f \perp \underline{d}$, then moving in the direction $\underline{d}$ does not change f and hence doesn't move us off the line tangent to the curve

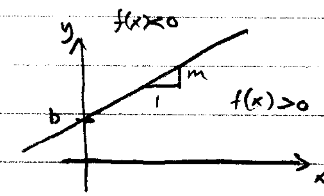
Importance of normal:

- It's perpendicular to (the tangent of) the curve
- It points toward the outside.

2D curves – implicit cont.

Representing lines in 2D implicit form.

We're all familiar with $y = mx + b$, or make that $mx - y + b = 0$.
so $f(x, y) = mx - y + b$.



note that $2f$ represents the same curve as f...

one problem w/ this scheme is that vertical lines are not possible. If we're willing to give up the uniqueness bestowed by having no coefficient in front of y we can generalize to the form

$$ax + by + c = 0 \quad \text{or} \quad \underline{u} \cdot \underline{x} + c = 0 \quad \text{for} \quad \underline{u} = (a, b) \text{ and } \underline{x} = (x, y)$$

This is nice because we can have $x + c = 0$ and $y + c = 0$ and everything in between.

$$f(\underline{x}) = \underline{u} \cdot \underline{x} + c \quad \Rightarrow \quad \nabla f(\underline{x}) = \underline{u}$$

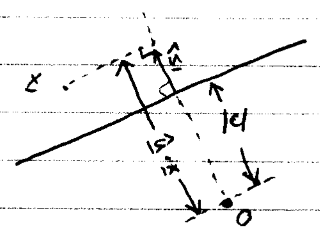
so another nice thing here is that $\underline{u}$ gives us the perpendicular.
If we require $\underline{u}$ to be a unit vector $\hat{\underline{u}}$ (ok since $\underline{u} = 0$ makes no sense) then:

- → c is the distance along $\hat{u}$ from the line to the origin
- → $f(\underline{x})$ is the distance from $\underline{x}$ to the line

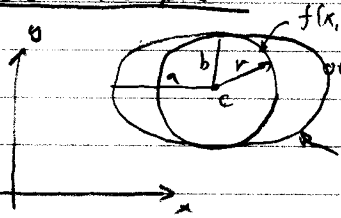
re-interpret equation as

$$f(\underline{x}) = \underbrace{\hat{\underline{u}} \cdot \underline{x}}_{\text{projection of } \underline{x} \text{ on } \underline{u}} - \underbrace{(-c)}_{\text{value of projection for pts. on the line}}$$

alternate form: $\hat{\underline{u}} \cdot (\underline{x} - \underline{p}) = 0$



One more quick example



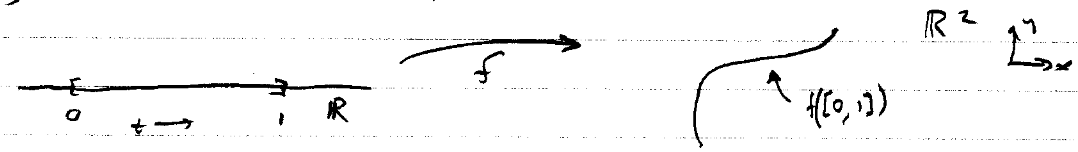
$$f(x, y) = (x - c_x)^2 + (y - c_y)^2 - r$$

or $f(x, y) = \left(\frac{x - c_x}{r}\right)^2 + \left(\frac{y - c_y}{r}\right)^2 - 1$

$$f(x, y) = \left(\frac{x - c_x}{a}\right)^2 + \left(\frac{y - c_x}{b}\right)^2 - 1$$

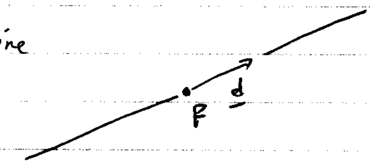
2D curves – parametric

Second major approach: explicit, or parametric, curves
now $f: \mathbb{R} \to \mathbb{R}^2$. The idea is that f maps some onto the plane, laying it out in some way:



the curve is the image of an interval (or all of $\mathbb{R}$) in $\mathbb{R}^2$.
i.e. $\{f(t) \mid t \in D\}$ — image of D under f.

→ e.g.: line

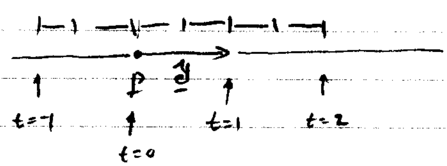


$L = \{\underline{p} + t\underline{d} \mid t \in \mathbb{R}\}$ $\quad [\underline{f}(t) = \underline{p} + t\underline{d}]$

This is more "explicit" than the implicit form because it directly gives us a way to compute points that lie on the line.

Note: a redundant representation (like the implicit form) but even more so: 4 numbers to represent 2 DOFs rather than just 3.

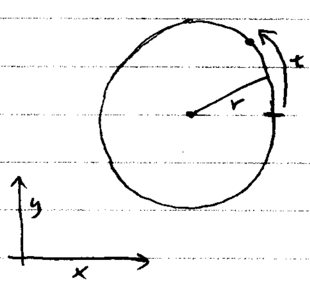
if $\underline{d}$ is unit length then t measures distance along the line:



"start at $\underline{p}$ and go a distance t in the direction $\underline{d}$"

We can represent any line in either of these two ways; the choice depends on convenience.

→ second e.g.: circle



t is an angle now.

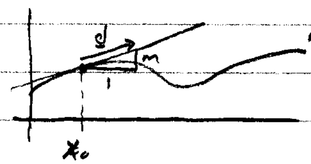
$\underline{f}(t) = (x_c + r\cos t,\ y_c + r\sin t)$

if we want one copy just set $D = [0, 2\pi)$.
how about the axis-aligned ellipse with radii a, b? $\quad f(t) = (x_c + a\cos t,\ y_c + b\sin t)$

2D curves — parametric (cont.)

Physical interpretation: particle moving with position $f(t)$ at time t.
thus usage of terms like "speed", "velocity", etc.

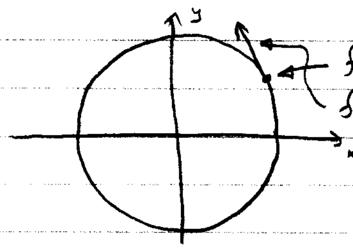
Computing tangents to parametric curves
- a tangent is an approximating line, like a derivative



$y = g(x)$, or, $f(t) = (t, g(t))$

- slope of tangent is $m = y'(t)$
- direction vector of tangent is $(1, m)$ — this is $f'(t)$
  - so parametric tangent at x_0 is $h(t) = f(x_0) + t f'(x_0)$
    and the tangent vector is the derivative

circle e.g. again

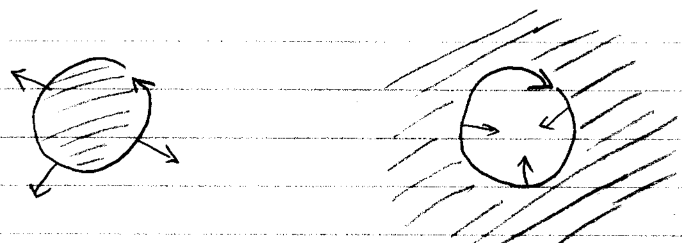


$f(t) = (\cos t, \sin t)$
$f'(t) = (-\sin t, \cos t)$

Note length of tangent vector depends on speed of parameterization
$f(t) = (\cos 2t, \sin 2t)$ gives length-2 tangent.

Often in graphics we want the normal rather than the tangent.
There is an arbitrary choice to make of which way the normal faces — that is, which side is inside and which is outside (the "orientation" of the curve).

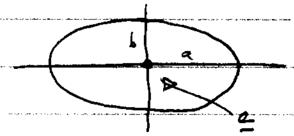
Convention: outside is to your right as you move along the curve
→ a counter-clockwise loop encloses the inside; a clockwise loop encloses a hole:



So if the tangent is (x, y) then the normal is $(y, -x)$.

2D Parametric Curves cont.

one last example:



$f(t) = \underline{c} + \begin{bmatrix} a \cos t \\ b \sin t \end{bmatrix}.$

3D Implicit surfaces

$f: \mathbb{R}^3 \to \mathbb{R}$ $f(\underline{x}) = 0$ generally defines a surface
(note: one implicit equation defines an object of dimension one lower than the space).

e.g. $\|x\| - r = 0$. a sphere. Just as in 2D ∇f is the normal vector.

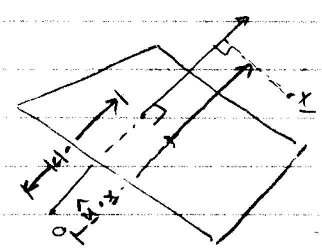
Implicit plane.

line: $f(\underline{x}) = \hat{\underline{a}} \cdot \underline{x} + c = 0$

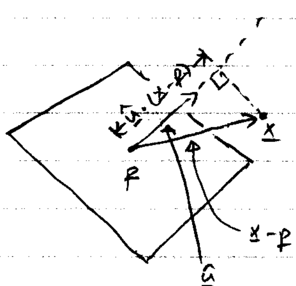
plane: $f(\underline{x}) = \hat{\underline{n}} \cdot \underline{x} + d = 0$ (same equation, one more dimension).

or, written out, $a x_1 + b x_2 + c x_3 + d = 0$

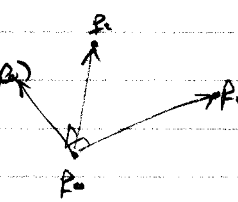
interpretation (same as line in 2D): distance from origin along normal vector is constant:



alternate interp: distance from point along normal is zero:



if you have 3 points, use one as $\underline{p}$ and cross the other two to get $\underline{n}$



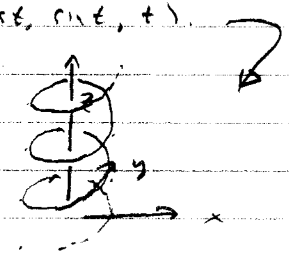
plane: $f(\underline{x}) = (\underline{x} - \underline{p}_0) \cdot \underline{n}$

3D parametric curves

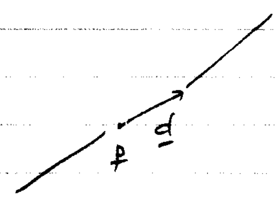
in 2D. $x_1 = f_1(t)$
$x_2 = f_2(t)$
we add $x_3 = f_3(t)$

$$\left.\begin{array}{l} x_1 = f_1(t) \\ x_2 = f_2(t) \\ x_3 = f_3(t) \end{array}\right\} \quad \underline{x} = \underline{f}(t) \qquad f: \mathbb{R} \to \mathbb{R}^3$$

e.g. in 2D circle was $\underline{f}(t) = (\cos t, \sin t)$
in 3D we can make a helix as: $\underline{f}(t) = (\cos t, \sin t, t)$.



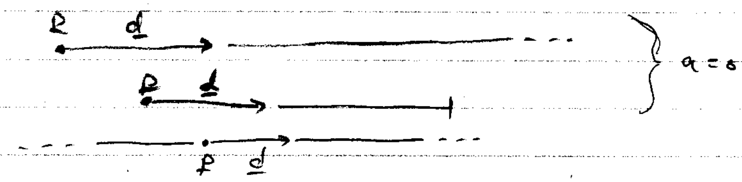
simplest parametric curve: line



exactly the same as 2D.

as in 2D, derivative $f'(t)$ is the tangent to the curve.

few more defns. that are used more often in 3D
$\underline{f}(t) = \underline{p} + t\underline{d}$
$\underline{f}([a, \infty))$ is a ray
$\underline{f}([a, b])$ is a line segment
$f((-\infty, \infty))$ is a line



note line through $\underline{p}_0$, $\underline{p}_1$ is $\underline{p}_0 + t(\underline{p}_1 - \underline{p}_0)$
segment between is that with $D = [0, 1]$.

3D parametric surfaces

now $f: \mathbb{R}^2 \to \mathbb{R}^3$
usually (s,t) or (u,v) — usually (x,y,z).

e.g. sphere example from earlier (in latitude-longitude coords)

$x = r \cos\phi \cos\theta$
$y = r \sin\phi \cos\theta$
$z = r \sin\theta$

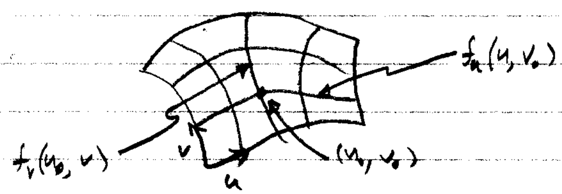
for parametric curves the derivative gave the tangent vector
What about in the 3D surface case?

3D parametric surfaces cont.

meaning of derivatives

A function $f: \mathbb{R}^2 \to \mathbb{R}^3$ has six derivatives $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \ldots$
$\frac{\partial x}{\partial v}, \ldots$

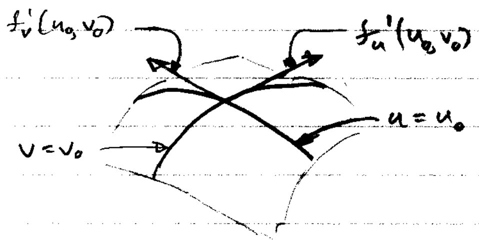
Notation: coordinate functions $f^x, f^y, f^z : \mathbb{R}^2 \to \mathbb{R}$
single-parameter functions $f_u(u; v_0)$, $f_v(u_0; v)$



Note that f_u and f_v define parametric curves (one for each value of the other variable)

This means that f_u' and f_v' are the tangents to the iso-parameter curves:

$f_u'(u_0, v_0)$ is the tangent to the $v = v_0$ curve
$f_v'(u_0, v_0)$ is the tangent to the $u = u_0$ curve.



These two vectors are tangent to the surface (since they're tangent to curves lying in the surface) and together they span the tangent plane at (u_0, v_0)

* → The normal to the surface is $\perp$ to this plane, so we can get it from $f_u' \times f_v'$.

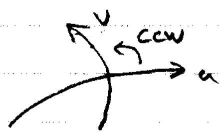
Note that the normal is not as nicely behaved as the tangents. It is not a linear function of f and we'll need to treat it carefully when we get to transformations of geometry.

3D parametric surfaces cont.

orientation - which side is outside, which is inside?

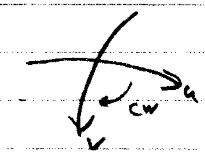
for implicit surfaces this was obvious: define based on sign$(f(\underline{x}))$, and we used the convention that the $f(\underline{x}) > 0$ side is outside. Hence normal ∇f points towards the outside.

for parametric surfaces its more subtle. Looking at a point on the surface you can draw the isoparametric curves. From one side they will look like:



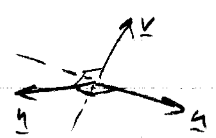
ccw = front

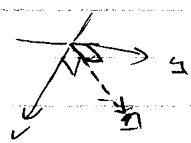
and from the other:



cw = back.

this is another of these arbitrary orientation choices, but in this case it is consistent with the right hand rule when computing the normal as $\underline{n} = f_u' \times f_v'$.





e.g. sphere

$$x = r\cos\phi\cos\theta \qquad x_\theta' = -r\cos\phi\sin\theta \qquad x_\phi' = r\sin\phi\cos\theta$$

$$y = r\sin\phi\cos\theta \qquad y_\theta' = -r\sin\phi\sin\theta \qquad y_\phi' = -r\cos\phi\cos\theta$$

$$z = r\sin\theta \qquad z_\theta' = r\cos\theta \qquad z_\phi' = 0.$$

$$\begin{bmatrix} x_\theta' \\ y_\theta' \\ z_\theta' \end{bmatrix} \times \begin{bmatrix} x_\phi' \\ y_\phi' \\ z_\phi' \end{bmatrix} = \begin{bmatrix} -(r\cos\theta)(-r\cos\phi\sin\theta) \\ (r\cos\theta)(r\sin\phi\sin\theta) \\ (-r\cos\phi\sin\theta)(-r\cos\phi\cos\theta) - (-r\sin\phi\sin\theta)(r\sin\phi\cos\theta) \end{bmatrix}$$

$$r^2\cos^2\phi\sin\theta\cos\theta + r^2\sin^2\phi\sin\theta\cos\theta$$

$$(r\cos\theta)(r\sin\theta)$$

note this is the direction (x, y, z) with length related to $r\sin\theta$.