CS 465 Homework Solution 6

due: Friday 28 October 2005

Problem 1: Direct spline manipulation

Hermite splines are cubic splines defined by point and tangent (that is, value and derivative) constraints at the two endpoints of each segment. More specifically, the following constraints define a Hermite segment:

$$\begin{aligned} \mathbf{p}(0) &= \mathbf{p}_0 \\ \mathbf{p}'(0) &= \mathbf{v}_0 \\ \mathbf{p}(1) &= \mathbf{p}_1 \\ \mathbf{p}'(1) &= \mathbf{v}_1 \end{aligned}$$

where $\mathbf{p}_0$ and $\mathbf{p}_1$ are the endpoints of the segment and $\mathbf{v}_0$ and $\mathbf{v}_1$ are the tangents (derivatives) of the curve at the endpoints. The resulting spline is defined as follows:

$$\mathbf{p}(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{v}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_1 \end{bmatrix}$$

You are working on a graphical editor for Hermite splines. The editor is meant to allow *direct manipulation*: rather than changing the control points directly, the user can click directly on the curve and drag it to a new shape. In particular, the user clicks on a point on a curve segment, which is $\mathbf{p}(t_c)$ for some parameter value t_c, and drags it to a new location $\mathbf{x}_c$. After this manipulation, $\mathbf{p}_{\text{new}}(t_c) = \mathbf{x}_c$.

In this problem you'll work out three different ways of making the spline pass through the user's desired point:

1. *By adjusting the tangents at the ends.* Make the spline pass through the new point while keeping the endpoints fixed by adjusting the tangents $\mathbf{v}_0$ and $\mathbf{v}_1$.

   (a) How many variables have been introduced and how many constraints? Is the solution expected to be unique?

   Solution

Two variables have been introduced: the value of each tangent at the endpoints. One constraint has been added: the position of the spline at time t_c. The solution will not be unique.

(b) Give an equation that constrains $\mathbf{v}_0$ and $\mathbf{v}_1$ so that the curve will pass through the point.

Solution

The constraint equation can be derived directly from the spline equation above. The values of v_0 and v_1 must be chosen so that the spline has the value x_c at time t_c:

$$\mathbf{x}_c = \begin{bmatrix} t_c^3 \\ t_c^2 \\ t_c \\ 1 \end{bmatrix}^T \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{v}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_1 \end{bmatrix}$$

Multiplying the right hand side out:

$$\mathbf{x}_c = \left(2\,{t_c}^3 - 3\,{t_c}^2 + 1\right)\mathbf{p}_0 + \left({t_c}^3 - 2\,{t_c}^2 + t_c\right)\mathbf{v}_0 + \left(-2\,{t_c}^3 + 3\,{t_c}^2\right)\mathbf{p}_1 + \left({t_c}^3 - {t_c}^2\right)\mathbf{v}_1$$

(c) Keep $\mathbf{v}_1$ fixed, and give an expression for $\mathbf{v}_0$ in terms of t_c and $\mathbf{x}_c$.

Solution

Solving the above equation for $\mathbf{v}_0$:

$$\mathbf{v}_0 = \frac{\left(2\,{t_c}^3 - 3\,{t_c}^2 + 1\right)\mathbf{p}_0 + \left(-2\,{t_c}^3 + 3\,{t_c}^2\right)\mathbf{p}_1 + \left({t_c}^3 - {t_c}^2\right)\mathbf{v}_1 - \mathbf{x}_c}{\left({t_c}^3 - 2\,{t_c}^2 + t_c\right)}$$

2. *By promoting the degree of the spline.* Make the spline pass through the new point while still maintaining the endpoint and tangent constraints by using a quartic spline that satisfies the Hermite constraints and the new constraint.

(a) How many variables have been introduced and how many constraints? Is the solution expected to be unique?

Solution

One constraint has been added: the position of the spline at time t_c. One constraint has been added: the coefficient of the quartic term in the new spline's polynomial. The solution will be unique.

(b) Give the matrix equation for the new spline, but it is OK to write the spline matrix as the inverse of another matrix. Please put the controls in the order $\mathbf{p}_0$, $\mathbf{v}_0$, $\mathbf{x}_c$, $\mathbf{p}_1$, $\mathbf{v}_1$.

Solution

First consider the constraints on the new spline. It must satisfy the four constraints of the cubic spline and the constraint that the position at t_c is x_c. To goal is to derive a 5-vector $\mathbf{a} = [a_4\ a_3\ a_2\ a_1\ a_0]$ such that:

$$\mathbf{p}(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

satisfies the contraints. Substituting the 5 constraints into the above equation gives a linear system whose solution is the required vector.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ t_c^4 & t_c^3 & t_c^2 & t_c & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{v}_0 \\ \mathbf{x}_c \\ \mathbf{p}_1 \\ \mathbf{v}_1 \end{bmatrix}$$

Solving for $\mathbf{a}$ and substituting into the above equation for the quartic spline:

$$\mathbf{p}(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ t_c^4 & t_c^3 & t_c^2 & t_c & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{v}_0 \\ \mathbf{x}_c \\ \mathbf{p}_1 \\ \mathbf{v}_1 \end{bmatrix}$$

(c) Show that your spline reduces to a cubic again if the user drags the point back exactly to where it started.

Solution

This part of the problem was more difficult than we anticipated so we chose to make the solution extra credit. No solution was penalized for this section.

To begin, some notation. Let $\mathbf{M}_3$ and $\mathbf{M}_4$ be the matrices in the definition of the cubic hermite spline defined at the top of this assignment and the quartic hermite spline discussed in this section respectively. Let $\mathbf{t}_i$ be a polynomial vector with highest degree i. For example, $\mathbf{t}_3 = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}$ and $\mathbf{t}_3(r) = \begin{bmatrix} r^3 & r^2 & r & 1 \end{bmatrix}$. Finally, let $\mathbf{x} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{v}_0 & \mathbf{p}_1 & \mathbf{v}_1 \end{bmatrix}^T$.

Since the proof will be long, a beginning summary is useful. Showing that the quartic spline reduces to a cubic requires demonstrating that the coefficient of the quartic term t^4 must be zero when then spline is in its original position.

This proof will proceed in two parts. First we will assemble a new representation of the quartic's coefficients in terms of the cubic's spline matrix. This requires rearranging the rows of the quartics spline matrix and expressing $\mathbf{x}_c$ in terms of its original position on the cubic spline. This representation will allow us to solve for the quartic coefficient a_4 when the spline is in its original position. The proof concludes by showing that its value must be zero.

To begin, note that:

$$\mathbf{M}_3^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

which at first seems mysterious, but is easily derived using the method in part (b) above for the cubic constraints. With this expression and by rearranging its rows, $\mathbf{M}_4$ can be written as a block matrix in terms of $\mathbf{M}_3$. Let $\mathbf{c} = \begin{bmatrix} 0 & 0 & 1 & 4 \end{bmatrix}^T$.

$$\mathbf{M}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ t_c^4 & t_c^3 & t_c^2 & t_c & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{c} & \mathbf{M}_3^{-1} \\ t_c^4 & \mathbf{t}_3 \end{bmatrix}^{-1} \tag{1}$$

If $\mathbf{x}_c$ has returned to its original position, it can be expressed as a point on the original cubic spline:

$$\mathbf{x}_c = \mathbf{t}_3(t_c)\mathbf{M}_3\mathbf{x}$$

Now the vector of control points for the quartic can also be rearranged and written as a block vector:

$$\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{v}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_1 \\ \mathbf{x}_c \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{t}_3(t_c)\mathbf{M}_3\mathbf{x} \end{bmatrix} \tag{2}$$

Putting together Equations (1) and (2) and using the definition of the coefficient vector $\mathbf{a}$ for the quartic spline from the previous section:

$$\mathbf{a} = \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} a_4 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{c} & \mathbf{M}_3^{-1} \\ t_c^4 & \mathbf{t}_3 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x} \\ \mathbf{t}_3(t_c)\mathbf{M}_3\mathbf{x} \end{bmatrix}$$

Or more clearly as a linear system:

$$\begin{bmatrix} \mathbf{c} & \mathbf{M}_3^{-1} \\ t_c^4 & \mathbf{t}_3 \end{bmatrix} \begin{bmatrix} a_4 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{t}_3(t_c)\mathbf{M}_3\mathbf{x} \end{bmatrix}$$

The final goal is to solve the above equation for a_4. Performing the multiplication generates two equations:

$$\begin{aligned} a_4\mathbf{c} + \mathbf{M}_3^{-1}\mathbf{a}_3 &= \mathbf{x} \\ a_4 t_c^4 + \mathbf{t}_3(t_c)\mathbf{a}_3 &= \mathbf{t}_3(t_c)\mathbf{M}_3\mathbf{x} \end{aligned}$$

Multipling the first by $-\mathbf{t}_3(t_c)\mathbf{M}_3$ and adding them together yields:

$$a_4(t_c^4 - \mathbf{t}_3(t_c)\mathbf{M}_3\mathbf{c}) = 0$$

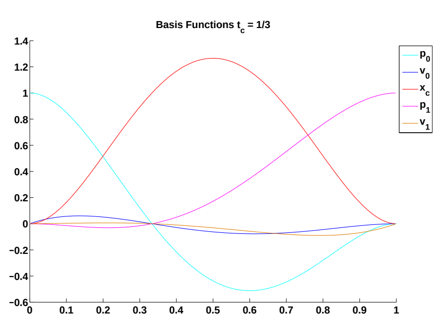
Recalling the definitions of $\mathbf{M}_3$ and $\mathbf{c}$, the expression can be simplified and factored:

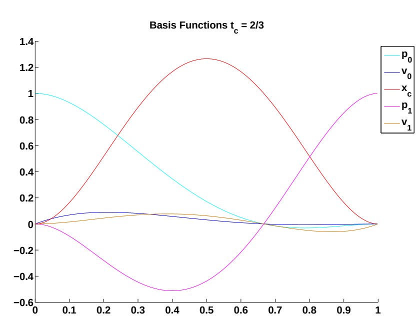
$$\begin{aligned} a_4(t_c^4 - \mathbf{t}_3(t_c) \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}) &= 0 \\ a_4(t_c^4 - \mathbf{t}_3(t_c) \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}) &= 0 \\ a_4(t_c^4 - 2t_c^3 + t_c^2) &= 0 \\ a_4 \cdot t_c^2(t_c - 1)^2 &= 0 \end{aligned}$$

Finally either $a_4 = 0$, $t_c = 0$ or $t_c = 1$. Each of the latter solutions are degenerate cases where the new constraint duplicates one of the original endpoint constraints and the foremost solution completes the proof.

(d) Plot the basis functions for your new spline when $t = \frac{1}{3}$ and $t = \frac{2}{3}$.

Solution





(e) What is the order of continuity of the curve at the manipulation point?

Solution

The new curve is a quartic polynomial and all polynomials are C^∞ continuous.

3. *By splitting the spline into two segments.* Make the spline pass through the new point by splitting it into two segments at the parameter t_c and then adjusting the endpoints of the new segments.

(a) How many variables have been introduced and how many constraints? Is the solution expected to be unique?

Solution

Four new variables have been added: the two endpoints and two tangents of the new curves. Two new constraints have been added: the positions of the two new endpoints should be at $\mathbf{x}_c$. The solution is not expected to be unique.

(b) Write expressions for the endpoints and tangents of the two curves that result from the split. Resolve ambiguity by keeping the curve continuous and keeping the tangent at the manipulation point unchanged.

Solution

The new endpoints of both curves should be $\mathbf{x}_c$ and the tangents of both curves should be equal to the tangent value of the original curve at t_c:

$$\begin{aligned}
\mathbf{p}'(t_c) &= \begin{bmatrix} 3t_c^2 \\ 2t_c \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{v}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_1 \end{bmatrix} \\
&= \left(6t_c^2 - 6t_c\right)\mathbf{p}_0 + \left(3t_c^2 - 4t_c + 1\right)\mathbf{v}_0 + \left(-6t_c^2 + 6t_c\right)\mathbf{p}_1 + \left(3t_c^2 - 2t_c\right)\mathbf{v}_1
\end{aligned}$$

Giving the following expressions for the two sub-splines:

$$\begin{aligned}
\mathbf{p}_1(t) &= \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{v}_0 \\ \mathbf{x}_c \\ \mathbf{p}'(t_c) \end{bmatrix} \\
\mathbf{p}_2(t) &= \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_c \\ \mathbf{p}'(t_c) \\ \mathbf{p}_1 \\ \mathbf{v}_1 \end{bmatrix}
\end{aligned}$$

(c) Spot-check that the curve coincides with the original if the user drags the point back exactly to where it started, by verifying that the points at $t = \frac{1}{2}$ on each of the two pieces are also on the original curve.

Solution

This part of the problem was more difficult than we anticipated so we chose to make the solution extra credit. No solution was penalized for this section.

The requirement is that the equivalent points on the original and new curves have the same value, or:

$$\begin{aligned} \mathbf{p}(\frac{t_c}{2}) &= \mathbf{p}_1(\frac{1}{2}) \\ \mathbf{p}(\frac{1+t_c}{2}) &= \mathbf{p}_2(\frac{1}{2}) \end{aligned}$$

when $\mathbf{x}_c = \mathbf{p}(t_c)$. Unfortunately, if the assignment was completed as written and the tangent at the new endpoint was kept fixed, it is actually impossible, in general, to show this because the subdivision would create new splines that will not, in general, conincide with the original. If, however, you did as some people discovered and rescaled the tangents at the new endpoints to account for the shorter new splines, the result could be shown algebraically by direct evaluation of the equations above. The rescaling required was the weigh the first spline's tangent values at the split by t_c and the second by $1 - t_c$.

(d) What is the order of continuity of the curve at the manipulation point?

Solution

The constraints only assure C^1 continuity.