2D Spline Curves

CS 4620 Lecture 26
Administration

• A4 due yesterday
  – Demos? Will get back to you

• PPA2 due on Monday

• CS 4621 has a project discussion today

• A5 out on Monday
Defining spline curves

• At the most general they are parametric curves

\[ S = \{ f(t) \mid t \in [0, N] \} \]

• For splines, \( f(t) \) is piecewise polynomial
  – for this lecture, the discontinuities are at the integers
Defining spline curves

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Defining spline curves

• Generally $f(t)$ is a piecewise polynomial
  – for this lecture, the discontinuities are at the integers
  – e.g., a cubic spline has the following form over $[k, k + 1]$:
    \[ x(t) = a_xt^3 + b_xt^2 + c_xt + d_x \]
    \[ y(t) = a_yt^3 + b_yt^2 + c_yt + d_y \]
  – Coefficients are different for every interval
Coordinate functions

2D spline
Coordinate functions

2D spline

coordinate function $x(t)$
Coordinate functions

2D spline

coordinate function $y(t)$

coordinate function $x(t)$

$0 \rightarrow 1 \rightarrow 2$

$t$
Coordinate functions

2D spline

coordinate function \( x(t) \)

coordinate function \( y(t) \)
Coordinate functions

2D spline

coordinate function $x(t)$

coordinate function $y(t)$
Coordinate functions
Coordinate functions

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Coordinate functions

2D spline

coordinate function $x(t)$

coordinate function $y(t)$
Coordinate functions

2D spline

coordinate function $y(t)$

coordinate function $x(t)$
Control of spline curves

• Specified by a sequence of controls (points or vectors)
• Shape is guided by control points (aka control polygon)
  – interpolating: passes through points
  – approximating: merely guided by points
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How splines depend on their controls

- Each coordinate is separate
  - the function $x(t)$ is determined solely by the $x$ coordinates of the control points
  - this means 1D, 2D, 3D, … curves are all really the same
Plan

1. Spline segments
   – how to define a polynomial on \([0,1]\)
   – …that has the properties you want
   – …and is easy to control

2. Spline curves
   – how to chain together lots of segments
   – …so that the whole curve has the properties you want
   – …and is easy to control

3. Refinement and evaluation
   – how to add detail to splines
   – how to approximate them with line segments
Spline Segments
Trivial example: piecewise linear

- This spline is just a polygon
  - control points are the vertices
- But we can derive it anyway as an illustration
- Each interval will be a linear function
  - $x(t) = at + b$
    - constraints are values at endpoints
  - $b = x_0$; $a = x_1 - x_0$
    - this is linear interpolation
Trivial example: piecewise linear

- Vector formulation

\[ x(t) = (x_1 - x_0)t + x_0 \]
\[ y(t) = (y_1 - y_0)t + y_0 \]
\[ f(t) = (p_1 - p_0)t + p_0 \]

- Matrix formulation

\[ f(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \]
Trivial example: piecewise linear

- Basis function formulation
  - regroup expression by \( p \) rather than \( t \)

\[
f(t) = (p_1 - p_0)t + p_0 = (1 - t)p_0 + tp_1
\]

- interpretation in matrix viewpoint

\[
f(t) = \begin{pmatrix} t & 1 \end{pmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}
\]
Trivial example: piecewise linear

- Vector blending formulation: “average of points”
  - blending functions: contribution of each point as $t$ changes

\[
\begin{align*}
  b_0(t) &= 1 - t \\
  b_1(t) &= t
\end{align*}
\]
Hermite splines

- Less trivial example
- Form of curve: piecewise cubic
- Constraints: endpoints and tangents (derivatives)
Hermite splines

- Solve constraints to find coefficients

\[ x(t) = at^3 + bt^2 + ct + d \]
\[ x'(t) = 3at^2 + 2bt + c \]
\[ x(0) = x_0 = d \]
\[ x(1) = x_1 = a + b + c + d \]
\[ x'(0) = x'_0 = c \]
\[ x'(1) = x'_1 = 3a + 2b + c \]

\[ d = x_0 \]
\[ c = x'_0 \]
\[ a = 2x_0 - 2x_1 + x'_0 + x'_1 \]
\[ b = -3x_0 + 3x_1 - 2x'_0 - x'_1 \]
Matrix form of spline

\[ f(t) = at^3 + bt^2 + ct + d \]

\[
\begin{bmatrix}
  t^3 & t^2 & t & 1
\end{bmatrix}
\begin{bmatrix}
  x & x & x & x & x \\
  x & x & x & x & x \\
  x & x & x & x & x \\
  x & x & x & x & x \\
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_2 \\
p_3 \\
\end{bmatrix}
\]

\[ f(t) = b_0(t)p_0 + b_1(t)p_1 + b_2(t)p_2 + b_3(t)p_3 \]
Matrix form of spline

\[ f(t) = at^3 + bt^2 + ct + d \]

\[ \begin{bmatrix} 
 t^3 & t^2 & t & 1 
 \end{bmatrix} \begin{bmatrix} 
 \times & \times & \times & \times \\
 \times & \times & \times & \times \\
 \times & \times & \times & \times \\
 \times & \times & \times & \times & \times 
\end{bmatrix} \begin{bmatrix} 
 p_0 \\
 p_1 \\
 p_2 \\
 p_3 
\end{bmatrix} \]

\[ f(t) = b_0(t)p_0 + b_1(t)p_1 + b_2(t)p_2 + b_3(t)p_3 \]
Matrix form of spline

\[ f(t) = at^3 + bt^2 + ct + d \]

\[
\begin{bmatrix}
  t^3 & t^2 & t & 1
\end{bmatrix}
\begin{bmatrix}
  x & x & x & x \\
  x & x & x & x \\
  x & x & x & x \\
  x & x & x & x
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_2 \\
p_3
\end{bmatrix}
\]

\[ f(t) = b_0(t)p_0 + b_1(t)p_1 + b_2(t)p_2 + b_3(t)p_3 \]
Hermite splines

- Matrix form is much simpler

\[ f(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ t_0 \\ t_1 \end{bmatrix} \]

- coefficients = rows
- basis functions = columns
Hermite splines

- Hermite blending functions
Hermite splines

- Hermite basis functions
Hermite splines

- Hermite basis functions
Hermite to Bézier

- Mixture of points and vectors is awkward
- Specify tangents as differences of points
Hermite to Bézier

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Hermite to Bézier

- Mixture of points and vectors is awkward
- Specify tangents as differences of points

\[ \begin{align*}
  p_0 & \quad q_0 & \quad q_1 & \quad q_2 & \quad p_1 \\
  t_0 & \quad q_0 & \quad q_1 & \quad q_2 & \quad -t_1
\end{align*} \]

- note derivative is defined as 3 times offset
  - reason is illustrated by linear case

I’m calling these points \( q \) just for this slide and the next one.
Hermite to Bézier

\[ p_0 = q_0 \]
\[ p_1 = q_3 \]
\[ t_0 = 3(q_1 - q_0) \]
\[ t_1 = 3(q_3 - q_2) \]

\[
\begin{bmatrix}
  p_0 \\
  p_1 \\
  v_0 \\
  v_1
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 \\
  -3 & 3 & 0 & 0 & 0 \\
  0 & 0 & -3 & 3 & 0
\end{bmatrix}
\begin{bmatrix}
  q_0 \\
  q_1 \\
  q_2 \\
  q_3
\end{bmatrix}
\]
Hermite to Bézier

\[ p_0 = q_0 \]
\[ p_1 = q_3 \]
\[ t_0 = 3(q_1 - q_0) \]
\[ t_1 = 3(q_3 - q_2) \]
Hermite to Bézier

\[
p_0 = q_0 \\
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t_0 = 3(q_1 - q_0) \\
t_1 = 3(q_3 - q_2)
\]

\[
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix} = \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 3 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3
\end{bmatrix}
\]
Bézier matrix

\[ f(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \]

– note that these are the Bernstein polynomials

\[ b_{n,k}(t) = \binom{n}{k} t^k (1 - t)^{n-k} \]

and that defines Bézier curves for any degree
Bézier basis