CS 4620 Assignment 5: Splines (Written)

out: Monday 2 November 2015

due: Friday 13 November 2015 (11:59pm) **Do this part alone.**

1. The deCastlejau algorithm splits a Bezier curve **B** into two curves: **L** and **R**. Prove that together the **L** and **R** curves compute the same original curve. (Hint: figure out what the equation of a point is at a location t in **L**. Prove that this is the point you would want it to be on **B**. The proof for the right curve **R** is similar and will not be needed.).

2. Extending Modified Béziers with Quintic Polynomials

   (a) Control points can be used to describe curved geometry. One particular curve type is the cubic Bézier. This type of curve is commonly used in graphics because it is intuitive to control and cubics are generally sufficient for the human visual system – humans can hardly distinguish between $\mathcal{C}^2$ continuity and higher levels of continuity.

   i. If we are representing a curve segment as a polynomial of degree n, how many control points do we need to specify?

   ii. We have a quintic Bézier (degree 5). How many control points do we need?

   iii. What is the minimum number of control points needed to model a parabola-shaped spline?

   (b) In this question we want to add additional control by upgrading our Bézier with the addition of certain properties. Specifically, we would like to modify our spline so that, in addition to having control over $f(0)$, $f(1)$, $f'(0)$, and $f'(1)$, we'd now add control over $f''(0)$ and $f''(1)$.

   To do this, assume we specify two additional control points p_4 and p_5. Define p_4 such that $f''(0)$ is proportional to $(p_4 - p_0)$ similar to how $f'(0)$ is proportional to $(p_1 - p_0)$ in our original derivation. In the original derivation of Béziers, we assumed a proportionality constant of 3 for convenience. Similarly, let the proportionality constant associated with $(p_4 - p_0)$ be 4 – that is, $f''(0) = 4(p_4 - p_0)$. Similarly, we want $f''(1) = 4(p_5 - p_3)$.

   i. Give a matrix A such that this new type of curve can be expressed as $f(u) = uAP$ where u is a vector of $[1, u, u^2, u^3, u^4, u^5]$ and P is a 6 by 2 matrix where rows correspond to control points.

(c) Plot the blending (basis) functions of this curve from $u = 0$ to $u = 1$ and include your plot in your response.

3. Catmull-Rom Splines

The Catmull-Rom spline is a member of the Cardinal spline family, with $\mathcal{C}^1$ continuity and control points that **lie directly on the spline**. Given a vector of control points, P_{CR}, the Catmull-Rom spline satisfies the matrix equation:

$$f_{CR}(u) = [u^3, u^2, u, 1] M_{CR} P_{CR}$$

$$\text{where } M_{CR} = \begin{pmatrix} -\tau & 2-\tau & \tau-2 & \tau \\ 2\tau & \tau-3 & 3-2\tau & -\tau \\ -\tau & 0 & \tau & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \tau \begin{pmatrix} -1 & 2/\tau - 1 & 1-2/\tau & 1 \\ 2 & 1-3/\tau & 3/\tau - 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1/\tau & 0 & 0 \end{pmatrix}$$

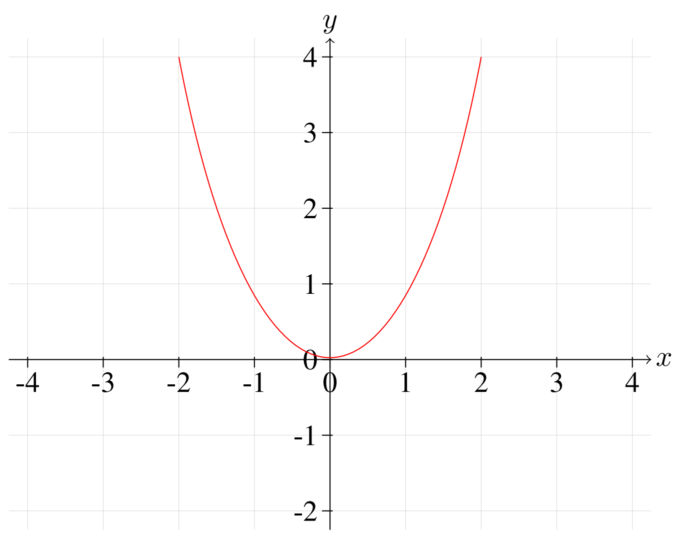
τ is a tension parameter chosen from the interval $(0, 1)$. When $\tau = 1/2$, we have

$$M_{CR} = \begin{pmatrix} -1/2 & 3/2 & -3/2 & 1/2 \\ 1 & -5/2 & 2 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

Draw a parabola (example: something that looks like $y = x^2$), and pretend that this is a 2-dimensional Catmull-Rom spline that has $\tau \approx 0.5$.

(a) What does the spline look like when τ is very small (i.e. $\tau \approx 0.2$)? Provide a sketch or describe your result.

(b) What does the spline look like when τ is very large (i.e. $\tau \approx 0.8$)? Provide a sketch or describe your result.



$\tau \approx 0.5$

(c) Catmull-Rom to Bézier

A Bézier spline segment has the following form:

$$f_{Bez}(u) = [u^3, u^2, u, 1] M_{Bez} P_{Bez}$$
$$\text{where } M_{Bez} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Converting a set of Catmull-Rom control points to a set of Bézier control points (which you can use to approximate the curve) is the same thing as solving for the matrix M such that

$$\text{if } P_{Bez} = M P_{CR}$$
$$\text{then } f_{Bez}(u) = f_{CR}(u) \text{ for all } u$$

Compute M with $\tau = 0.5$.

(d) Convert the 4 Catmull-Rom control points ($\tau = 0.5$) below into Bézier control points.

$$p_0 = (-5, 3)$$
$$p_1 = (0, 0)$$
$$p_2 = (1, 2)$$
$$p_3 = (3, 8)$$