CS4620/5620: Lecture 28

Splines

Announcements

• Practicum
  – This week (PPA2)
  – Next week (PPA3)
Defining spline curves

• At the most general they are parametric curves

\[ S = \{ p(t) \mid t \in [0, N] \} \]

• Generally \( p(t) \) is a piecewise polynomial
  – the discontinuities are at the integers

\[ x(t) = a_xt^3 + b_xt^2 + c_xt + d_x \]
\[ y(t) = a_yt^3 + b_yt^2 + c_yt + d_y \]
  – Coefficients are different for every interval
Control of spline curves

• Specified by a sequence of control points
• Shape is guided by control points (aka control polygon)
  – interpolating: passes through points
  – approximating: merely guided by points

Trivial example: piecewise linear

• This spline is just a polygon
  – control points are the vertices
• But we can derive it anyway as an illustration
• Each interval will be a linear function
  – \( x(t) = at + b \)
  – constraints are values at endpoints
  – \( b = x_0 \); \( a = x_1 - x_0 \)
  – this is linear interpolation
Trivial example: piecewise linear

• Basis function formulation
  – regroup expression by \( p \) rather than \( t \)

\[
p(t) = (p_1 - p_0)t + p_0 \\
= (1 - t)p_0 + tp_1
\]

– interpretation in matrix viewpoint

\[
p(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}
\]

Trivial example: piecewise linear

• Vector blending formulation: “average of points”
  – blending functions: contribution of each point as \( t \) changes

![Diagram showing blending functions](image)
Trivial example: piecewise linear

- Basis function formulation: “function times point”
  - basis functions: contribution of each point as $t$ changes
- can think of them as blending functions glued together
- this is just like a reconstruction filter!

Hermite splines

- Less trivial example
- Form of curve: piecewise cubic
  - How many constraints?
- Constraints: endpoints and tangents (derivatives)
Defining spline curves

• Generally $p(t)$ is a piecewise polynomial
  – the discontinuities are at the integers
  – e.g., a cubic spline has the following form over $[k, k + 1]$:

\[
x(t) = a_x t^3 + b_x t^2 + c_x t + d_x \\
y(t) = a_y t^3 + b_y t^2 + c_y t + d_y
\]

– Coefficients are different for every interval

Hermite splines

• Solve constraints to find coefficients

\[
x(t) = at^3 + bt^2 + ct + d \\
x'(t) = 3at^2 + 2bt + c \\
x(0) = x_0 = d \\
x(1) = x_1 = a + b + c + d \\
x'(0) = x'_0 = c \\
x'(1) = x'_1 = 3a + 2b + c \\
d = x_0 \\
c = x'_0 \\
a = 2x_0 - 2x_1 + x'_0 + x'_1 \\
b = -3x_0 + 3x_1 - 2x'_0 - x'_1
\]
Hermite splines

- Matrix form is much simpler

\[ p(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix} \]

- coefficients = rows
- basis functions = columns

Matrix form of spline

\[ p(t) = at^3 + bt^2 + ct + d \]

\[ \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \]

\[ p(t) = b_0(t)p_0 + b_1(t)p_1 + b_2(t)p_2 + b_3(t)p_3 \]
Hermite splines

• Matrix form is much simpler

\[
p(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix}
\]

- coefficients = rows
- basis functions = columns

Hermite splines

• Hermite basis functions
Longer Hermite splines

- Can only do so much with one Hermite spline
- Can use these splines as segments of a longer curve
  - curve from $t = 0$ to $t = 1$ defined by first segment
  - curve from $t = 1$ to $t = 2$ defined by second segment
- To avoid discontinuity, match derivatives at junctions
  - this produces a $C^1$ curve
Hermite to Bézier

- Mixture of points and vectors is awkward
- Specify tangents as differences of points

\[ \begin{align*}
 p_0 &= q_0 \\
 p_1 &= q_3 \\
 v_0 &= 3(q_1 - q_0) \\
 v_1 &= 3(q_3 - q_2)
\end{align*} \]

\[
\begin{bmatrix}
 a \\
 b \\
 c \\
 d
\end{bmatrix} =
\begin{bmatrix}
 -1 & 3 & -3 & 1 \\
 3 & -6 & 3 & 0 \\
 -3 & 3 & 0 & 0 \\
 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
 q_0 \\
 q_1 \\
 q_2 \\
 q_3
\end{bmatrix}
\]
Bézier matrix

\[ p(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \]

–note that these are the Bernstein polynomials

\[ C(n,k) \ t^k \ (1 - t)^{n-k} \]

and that defines Bézier curves for any degree

Bézier basis