CS4620/5620

Affine and 3D Transformations

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Announcements

- Updated schedule on course web page
  - 2 Prelim days finalized and posted
    - Oct 11, Nov 29
  - No final exam, final project will be due in the week of finals
- First homework will be out this weekend

Linear transformations using matrices

- One way to define a transformation is by matrix multiplication:

$$T(\mathbf{v}) = M\mathbf{v}$$

- Such transformations are *linear*, which is to say:

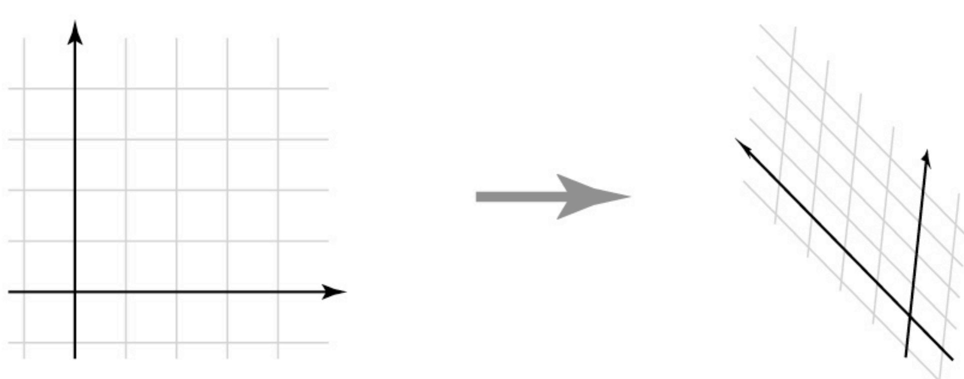
$$T(a\mathbf{u} + \mathbf{v}) = aT(\mathbf{u}) + T(\mathbf{v})$$

(and in fact all linear transformations can be written this way)

- Translation cannot be represented by linear transforms

Affine transformations

- The set of transformations we are interested in is known as the "affine" transformations
  - straight lines preserved; parallel lines preserved
  - ratios of lengths along lines preserved (midpoints preserved)



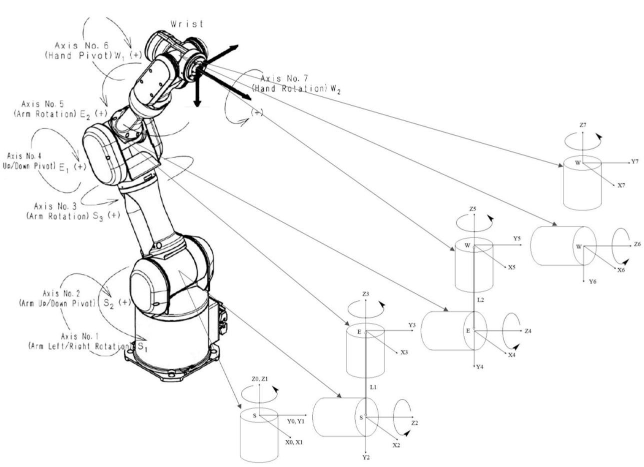
Transforming points and vectors

- Homogeneous coords. lets us handle points and vectors
  - just put 0 rather than 1 in the last place

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} M\mathbf{p} + \mathbf{t} \\ 1 \end{bmatrix} \quad \begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = \begin{bmatrix} M\mathbf{v} \\ 0 \end{bmatrix}$$

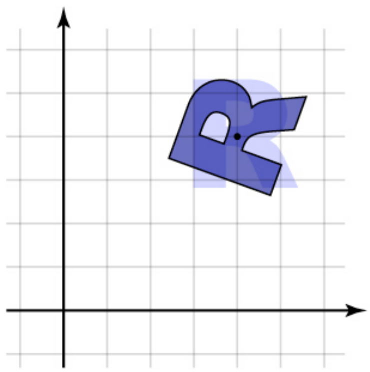
Coordinate Systems

- Bases
  - Expressing vectors in different bases
- Motivation
  - Global coordinate system
  - Local coordinate system



Composing to change axes

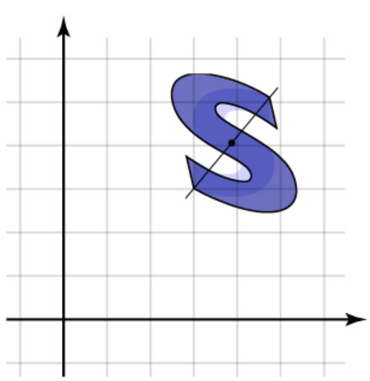
- Want to rotate about a particular point
  - could work out formulas directly…
- Know how to rotate about the origin
  - so translate that point to the origin



$$M = T^{-1}RT$$

Composing to change axes

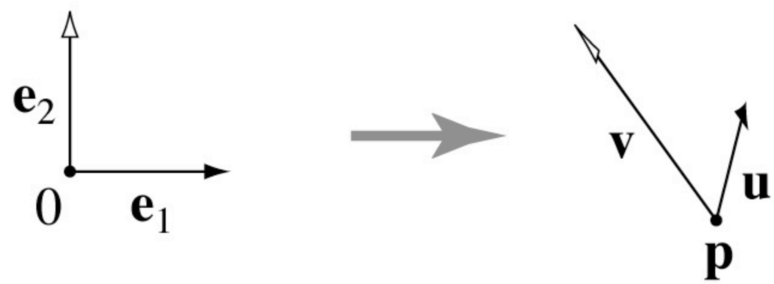
- Want to scale along a particular axis and point
- Know how to scale along the y axis at the origin
  - so translate to the origin and rotate to align axes



$$M = T^{-1}R^{-1}SRT$$

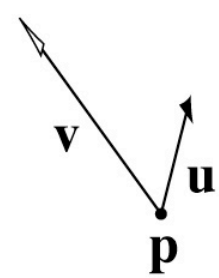
Another way of thinking about this

- Affine change of coordinates



Affine change of coordinates

- Coordinate frame: point plus basis
- Need to change representation of point from one basis to another
- Canonical: origin (0,0) w/ axes e1, e2

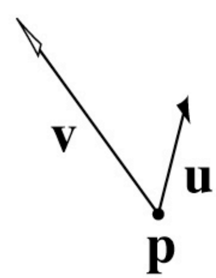


- "Frame to canonical" transformation
- Seems backward but worth thinking about

On the Board

Affine change of coordinates

- Coordinate frame: point plus basis
- Need to change representation of point from one basis to another
- Canonical: origin (0,0) w/ axes e1, e2

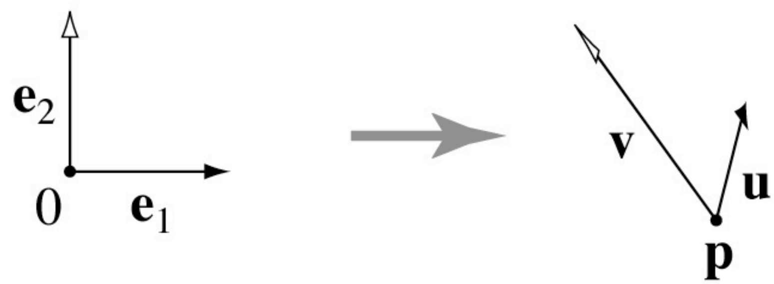


- "Frame to canonical" matrix has frame in columns
  - takes points represented in frame
  - represents them in canonical basis

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$

Another way of thinking about this

- Affine change of coordinates
  - Six degrees of freedom



$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$

Affine change of coordinates

- When we move an object to the origin to apply a transformation, we are really changing coordinates
  - the transformation is easy to express in object's frame
  - so define it there and transform it

$$T_e = F T_F F^{-1}$$

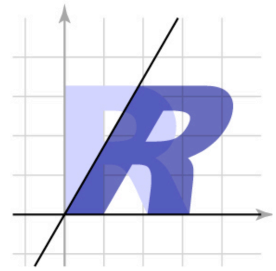
  - T_e is the transformation expressed wrt. e_1, e_2 (canonical, world)
  - T_F is the transformation expressed in natural (local) frame
  - F is the frame-to-canonical matrix [u v p]

- This is a similarity matrix

Affine change of coordinates

- A new way to "read off" the matrix
  - e.g. shear from earlier
  - can look at picture, see effect on basis vectors, write down matrix

$$\begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Also an easy way to construct transforms
  - e. g. scale by 2 across direction (1,2)

Coordinate frame summary

- Frame = point plus basis
- Frame matrix (frame-to-canonical) is

$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$

- Move points to and from frame by multiplying with *F*

$$p_e = F p_F \quad p_F = F^{-1} p_e$$

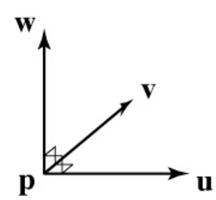
- Move transformations

$$T_e = F T_F F^{-1} \quad T_F = F^{-1} T_e F$$

Orthonormal frames in 3D

- Useful tools for constructing transformations
- Recall rigid motions
  - affine transforms with pure rotation
  - columns (and rows) form right-handed ONB
    - that is, an **o**rtho**n**ormal **b**asis

$$F = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$



$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

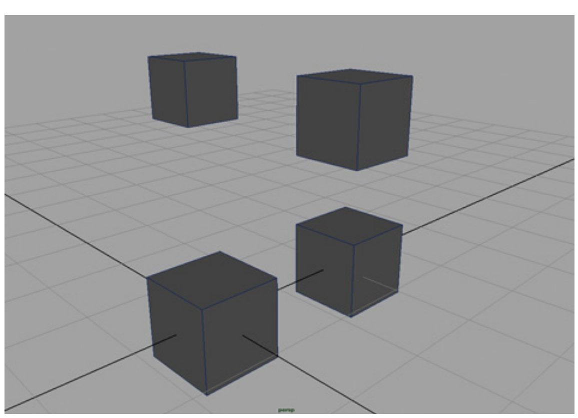
Fun and important aside: How to build 3D frames

- Given a vector **a,** define a frame with one axis parallel to **a**
  - Given a secondary vector **b**
    - The **u** axis should be parallel to **a**, the **u**–**v** plane should contain **b**
      - » **u** = **a** / ||**a**||
      - » **w** = **u** x **b**; **w** = **w** / ||w||
      - » **v** = **w** x **u**
- Given just a vector **a**
  - The **u** axis should be parallel to **a**; don't care about orientation about that axis
    - Same process but choose arbitrary **b** first
    - Good choice is not near **a**: e.g. set smallest entry to 1

3D Transformations

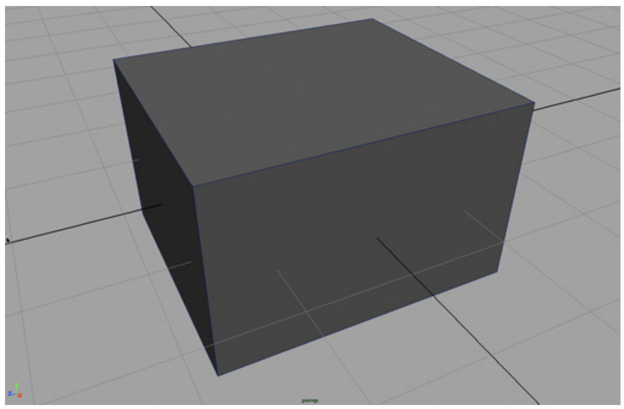
Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



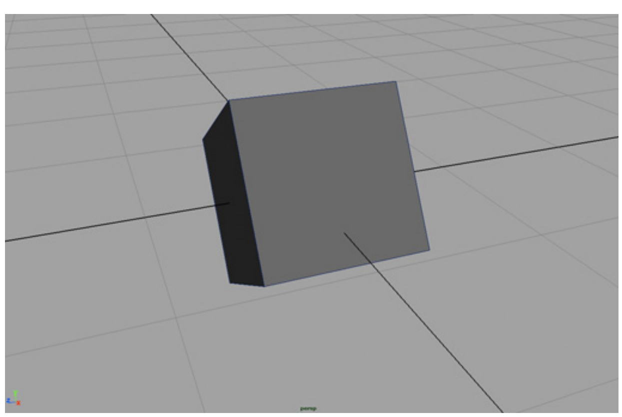
Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



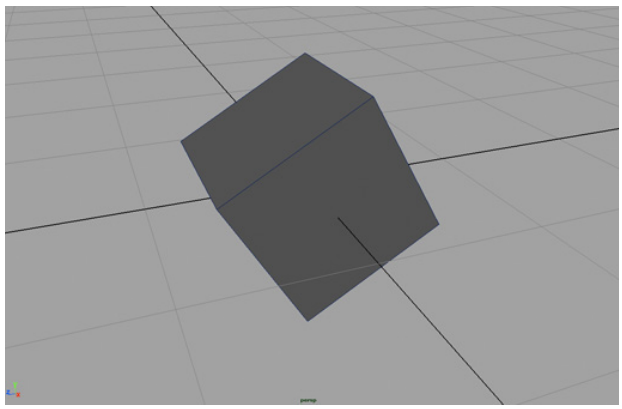
Rotation about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Rotation about *x* axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Rotation about y axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$