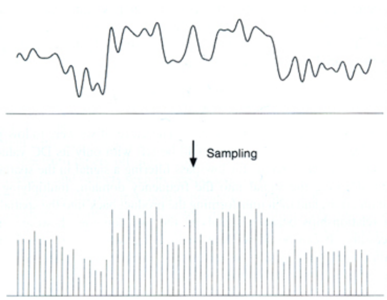
Sampling and reconstruction

CS 4620

Shirley and Marschner, 3rd ed
Chapter 9 "Signal Processing"

Sampled representations

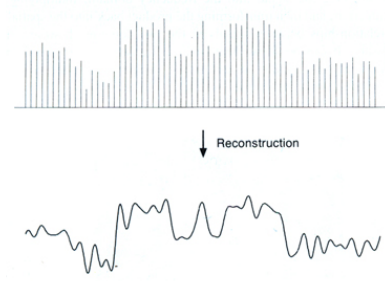
- How to store and compute with continuous functions?
- Common scheme for representation: samples
  - write down the function's values at many points



[FvDFH fig.14.14b / Wolberg]

Reconstruction

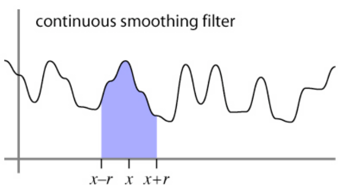
- Making samples back into a continuous function
  - for output (need realizable method)
  - for analysis or processing (need mathematical method)
  - amounts to "guessing" what the function did in between

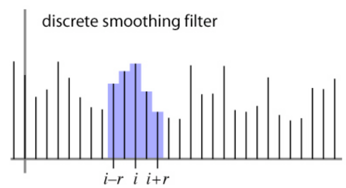


[FvDFH fig.14.14b / Wolberg]

Filtering

- Processing done on a function
  - can be executed in continuous form (e.g. analog circuit)
  - but can also be executed using sampled representation
- Simple example: smoothing by averaging



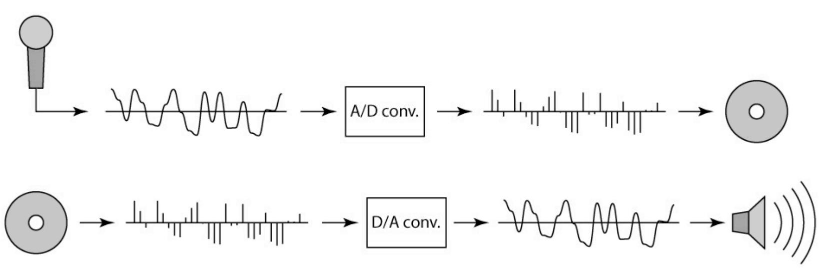


Roots of sampling

- Nyquist 1928; Shannon 1949
  - famous results in information theory
- 1940s: first practical uses in telecommunications
- 1960s: first digital audio systems
- 1970s: commercialization of digital audio
- 1982: introduction of the Compact Disc
  - the first high-profile consumer application
- This is why all the terminology has a communications or audio "flavor"
  - early applications are 1D; for us 2D (images) is important

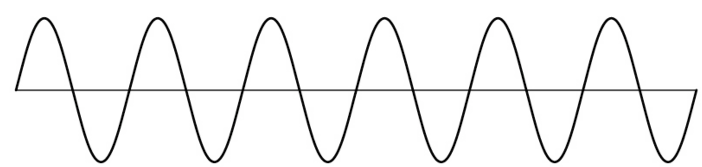
Sampling in digital audio

- Recording: sound to analog to samples to disc
- Playback: disc to samples to analog to sound again
  - how can we be sure we are filling in the gaps correctly?



Undersampling

- What if we "missed" things between the samples?
- Simple example: undersampling a sine wave
  - unsurprising result: information is lost
  - surprising result: indistinguishable from lower frequency
  - also was always indistinguishable from higher frequencies
  - *aliasing*: signals "traveling in disguise" as other frequencies



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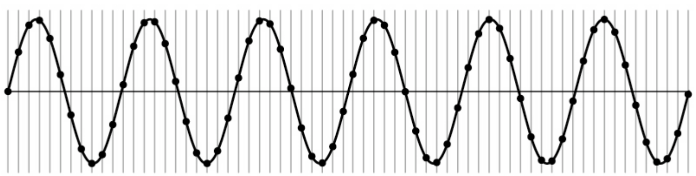

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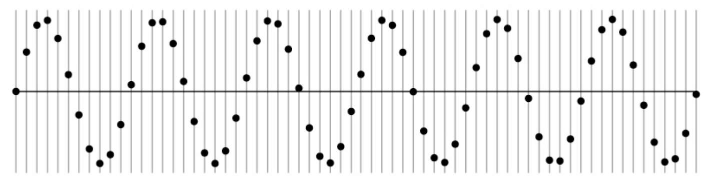

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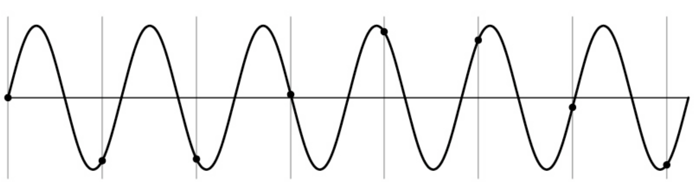

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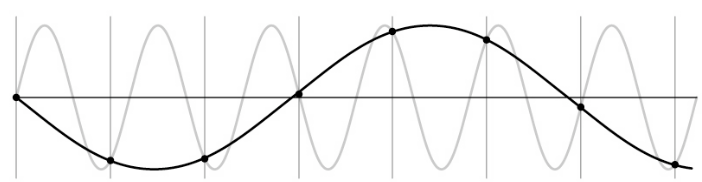

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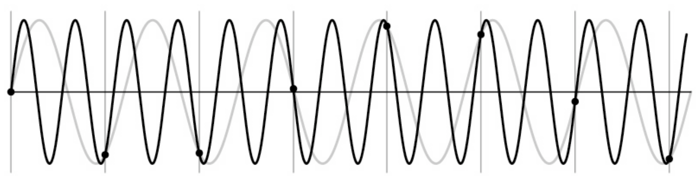
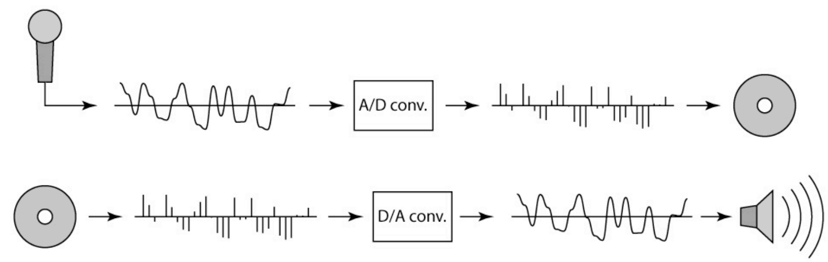


Preventing aliasing

- Introduce lowpass filters:
  - remove high frequencies leaving only safe, low frequencies
  - choose lowest frequency in reconstruction (disambiguate)



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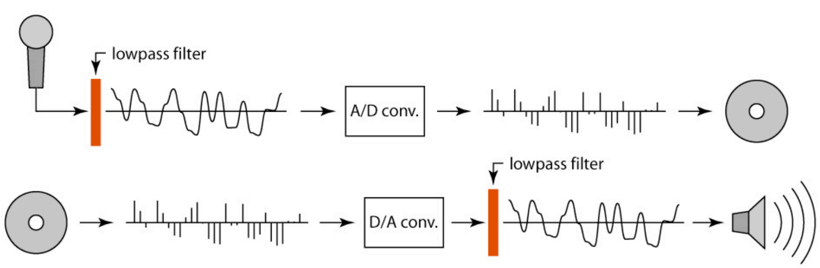
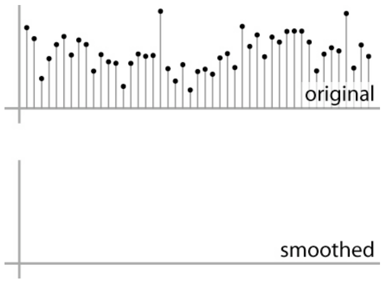


Linear filtering: a key idea

- Transformations on signals; e.g.:
  - bass/treble controls on stereo
  - blurring/sharpening operations in image editing
  - smoothing/noise reduction in tracking
- Key properties
  - linearity: filter(f + g) = filter(f) + filter(g)
  - shift invariance: behavior invariant to shifting the input
    - delaying an audio signal
    - sliding an image around
- Can be modeled mathematically by *convolution*

Convolution warm-up

- basic idea: define a new function by averaging over a sliding window
- a simple example to start off: smoothing



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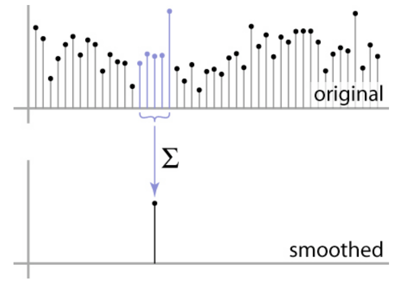

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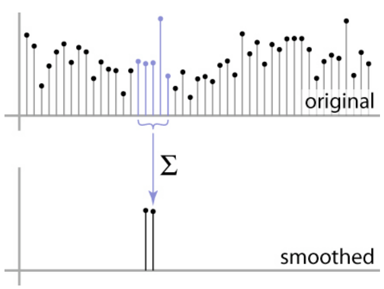

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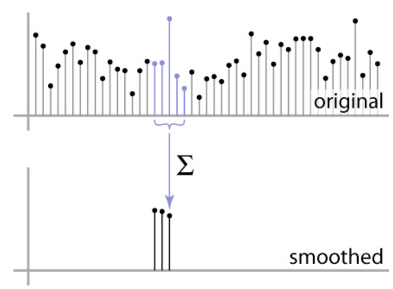

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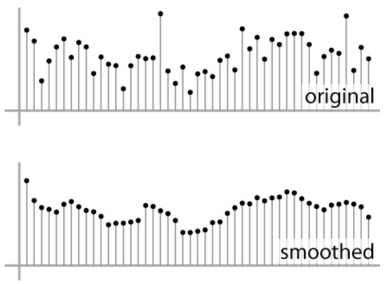


Convolution warm-up

- Same moving average operation, expressed mathematically:

$$b_{\text{smooth}}[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} b[j]$$

Discrete convolution

- Simple averaging:

$$b_{\text{smooth}}[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} b[j]$$

  every sample gets the same weight

- Convolution: same idea but with *weighted* average

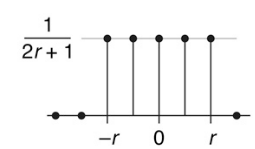
$$(a \star b)[i] = \sum_{j} a[j] b[i-j]$$

  each sample gets its own weight (normally zero far away)

- This is all convolution is: it is a **moving weighted average**

Filters

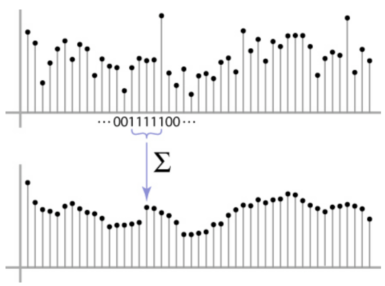
- Sequence of weights *a*[*j*] is called a *filter*
- Filter is nonzero over its *region of support*
  - usually centered on zero: support radius *r*
- Filter is *normalized* so that it sums to 1.0
  - this makes for a weighted average, not just any old weighted sum
- Most filters are symmetric about 0
  - since for images we usually want to treat left and right the same



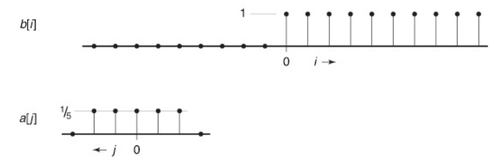
a box filter

Convolution and filtering

- Can express sliding average as convolution with a *box filter*
- a_{box} = [..., 0, 1, 1, 1, 1, 1, 0, ...]



Example: box and step



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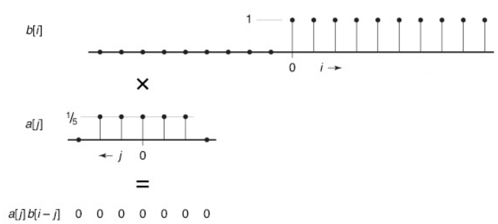

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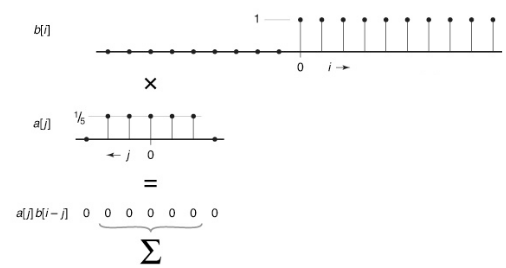

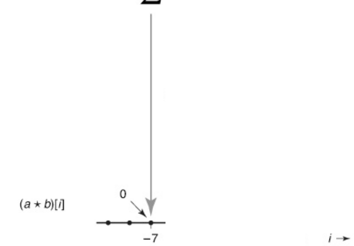

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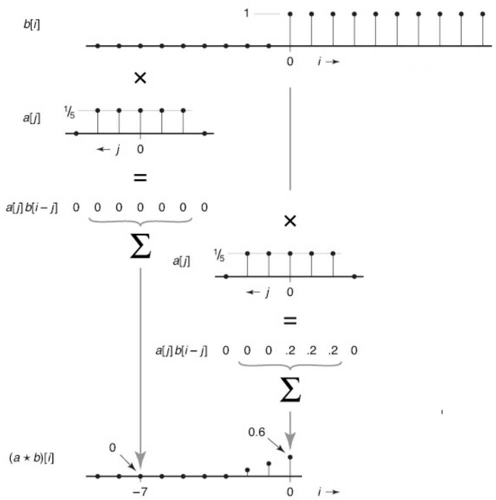

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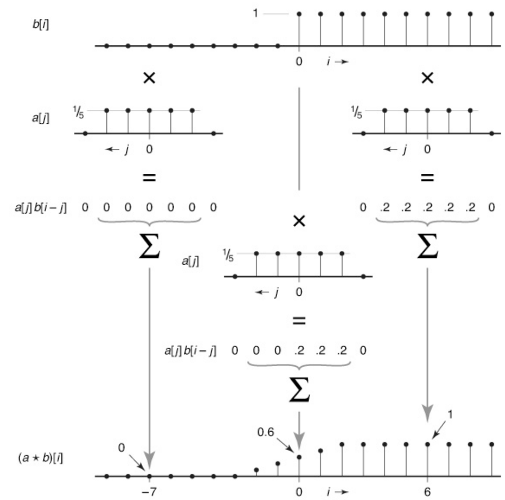
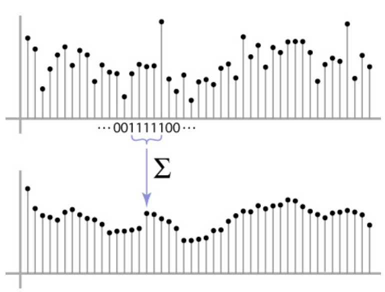


Convolution and filtering

- Convolution applies with any sequence of weights
- Example: Bell curve (Gaussian-like) […, 1, 4, 6, 4, 1, …]/16



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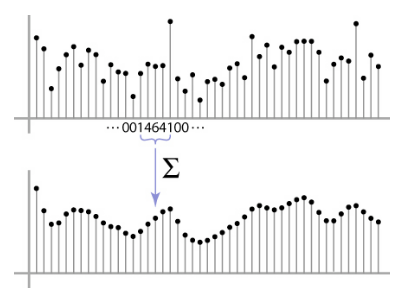


And in pseudocode…

function convolve(sequence a, sequence b, int r, int i)
  $s = 0$
  for $j = -r$ to r
    $s = s + a[j]b[i-j]$
  return s

Discrete convolution

- Notation: $b = c \star a$
- Convolution is a multiplication-like operation
  - commutative $a \star b = b \star a$
  - associative $a \star (b \star c) = (a \star b) \star c$
  - distributes over addition $a \star (b + c) = a \star b + a \star c$
  - scalars factor out $\alpha a \star b = a \star \alpha b = \alpha(a \star b)$
  - identity: unit impulse e = […, 0, 0, 1, 0, 0, …]
    $a \star e = a$
- Conceptually no distinction between filter and signal

Discrete filtering in 2D

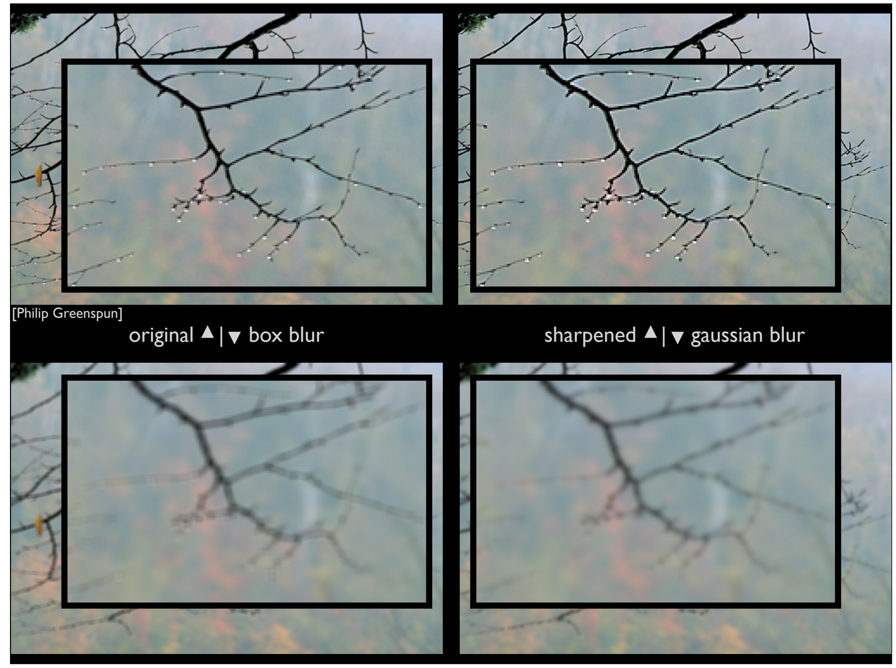
- Same equation, one more index

$$(a \star b)[i, j] = \sum_{i', j'} a[i', j']b[i - i', j - j']$$

  now the filter is a rectangle you slide around over a grid of numbers
- Commonly applied to images
  - blurring (using box, using gaussian, …)
  - sharpening (impulse minus blur)
- Usefulness of associativity
  - often apply several filters one after another: $(((a * b_1) * b_2) * b_3)$
  - this is equivalent to applying one filter: a * $(b_1 * b_2 * b_3)$

And in pseudocode…

function convolve2d(filter2d a, filter2d b, int i, int j)
$s = 0$
$r = a$.radius
for $i' = -r$ to r **do**
  for $j' = -r$ to r **do**
    $s = s + a[i'][j']b[i - i'][j - j']$
return s



Optimization: separable filters

- basic alg. is $O(r^2)$: large filters get expensive fast!
- definition: $a_2(x,y)$ is *separable* if it can be written as:

$$a_2[i,j] = a_1[i]a_1[j]$$

this is a useful property for filters because it allows factoring:

$$\begin{aligned}(a_2 \star b)[i,j] &= \sum_{i'}\sum_{j'} a_2[i',j']b[i-i',j-j'] \\ &= \sum_{i'}\sum_{j'} a_1[i']a_1[j']b[i-i',j-j'] \\ &= \sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j']b[i-i',j-j']\right)\end{aligned}$$

Separable filtering

$$a_2[i,j] = a_1[i]a_1[j]$$

1	4	6	4	1
4	16	24	16	4
6	24	36	24	6
4	16	24	16	4
1	4	6	4	1

0	0	0	0	0
0	0	0	0	0
1	4	6	4	1
0	0	0	0	0
0	0	0	0	0

0	0	1	0	0
0	0	4	0	0
0	0	6	0	0
0	0	4	0	0
0	0	1	0	0

first, convolve with this

$$\sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j']b[i-i',j-j']\right)$$

Separable filtering

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0	0	1	0	0
0	0	4	0	0
0	0	6	0	0
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0	0	1	0	0

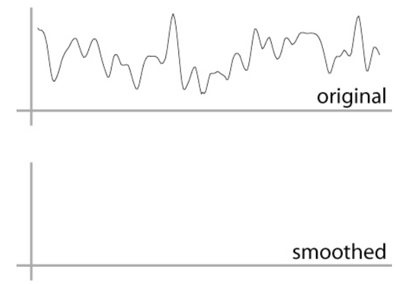
second, convolve with this

first, convolve with this

$$\sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j'] b[i-i', j-j'] \right)$$

Continuous convolution: warm-up

- Can apply sliding-window average to a continuous function just as well
  - output is continuous
  - integration replaces summation



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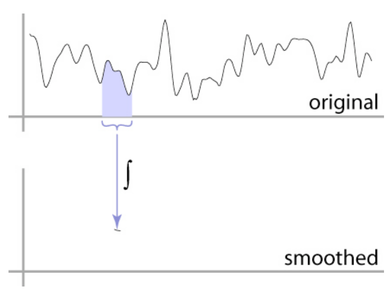

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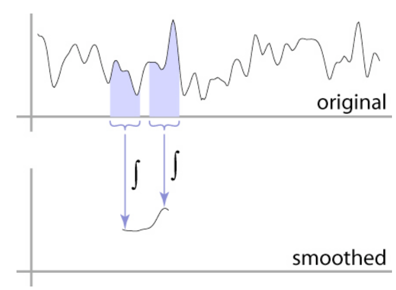

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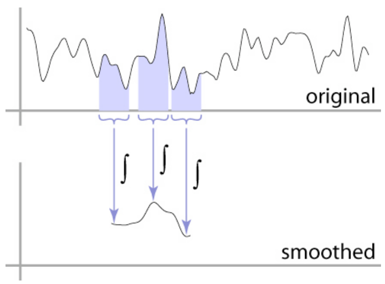

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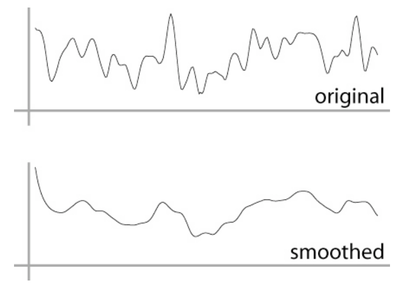


Continuous convolution

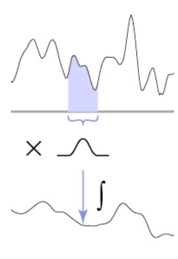
- Sliding average expressed mathematically:

$$g_{\text{smooth}}(x) = \frac{1}{2r} \int_{x-r}^{x+r} g(t) dt$$

  - note difference in normalization (only for box)
- Convolution just adds weights

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

  - weighting is now by a function
  - weighted integral is like weighted average
  - again bounds are set by support of *f(x)*



One more convolution

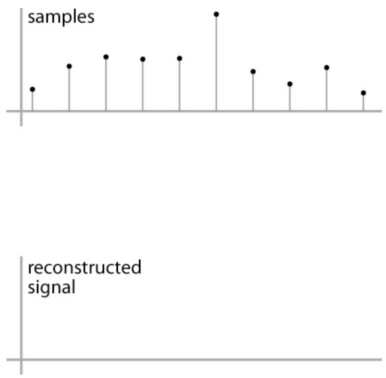
- Continuous–discrete convolution

$$(a \star f)(x) = \sum_i a[i] f(x-i)$$

$$(a \star f)(x, y) = \sum_{i,j} a[i,j] f(x-i, y-j)$$

  - used for reconstruction and resampling

Continuous-discrete convolution



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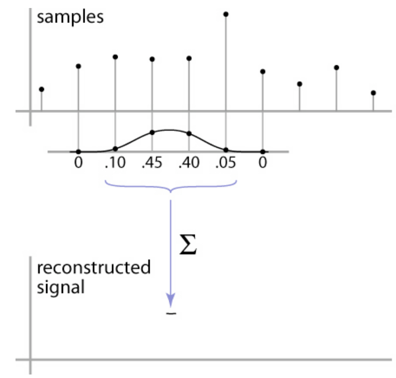

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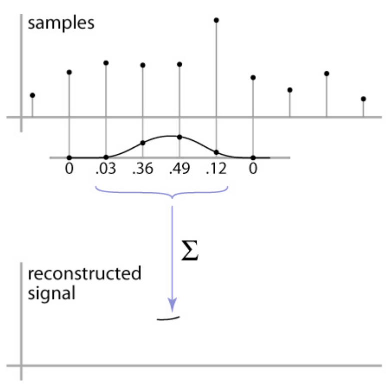

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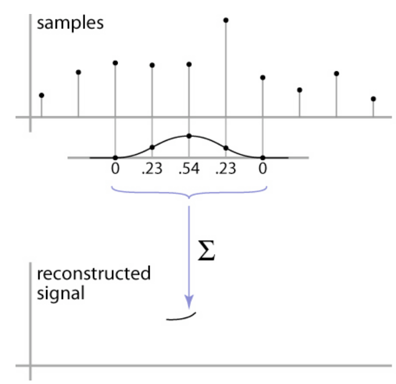

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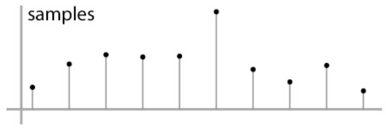
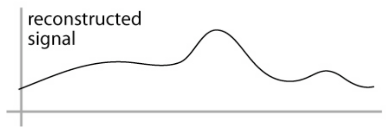


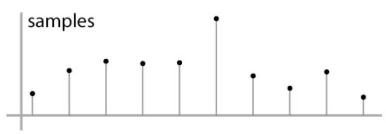


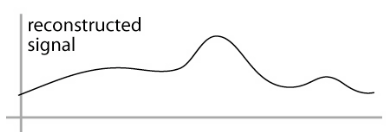
Resampling

- Changing the sample rate
  in images, this is enlarging and reducing
- Creating more samples:
  increasing the sample rate
  "upsampling"
  "enlarging"
- Ending up with fewer samples:
  decreasing the sample rate
  "downsampling"
  "reducing"

Resampling

- Reconstruction creates a continuous function
  forget its origins, go ahead and sample it





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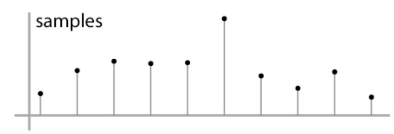
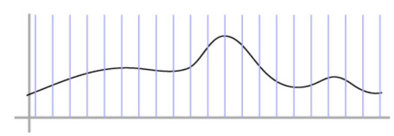
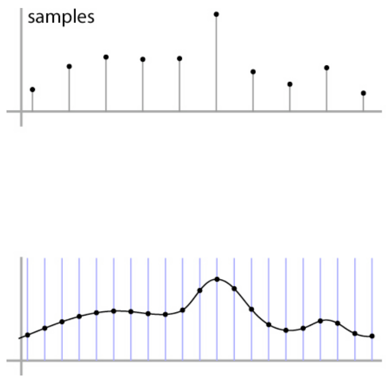




Resampling

- Reconstruction creates a continuous function

  forget its origins, go ahead and sample it



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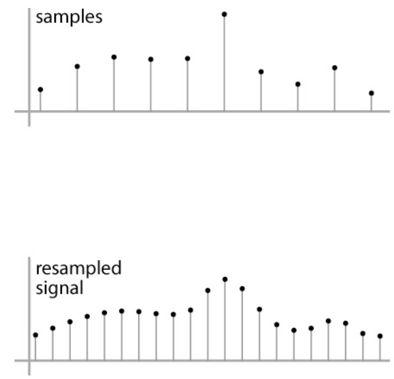

Resampling

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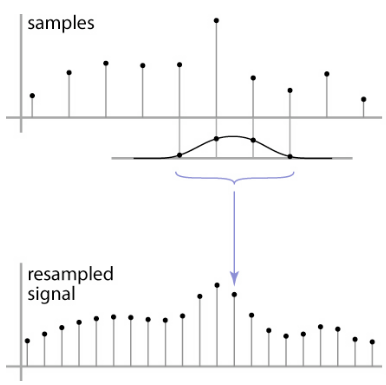


And in pseudocode…

function reconstruct(sequence a, filter f, real x)
$s = 0$
$r = f.\text{radius}$
for $i = \lceil x - r \rceil$ to $\lfloor x + r \rfloor$ **do**
    $s = s + a[i] f(x - i)$
return s

Cont.–disc. convolution in 2D

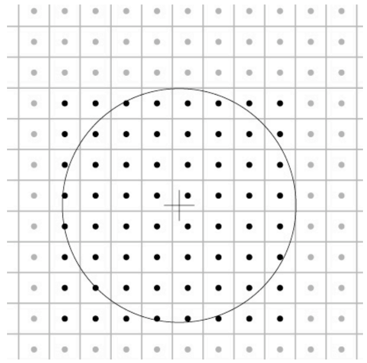
- same convolution—just two variables now

$$(a \star f)(x, y) = \sum_{i,j} a[i, j] f(x - i, y - j)$$

loop over nearby pixels,
average using filter weight

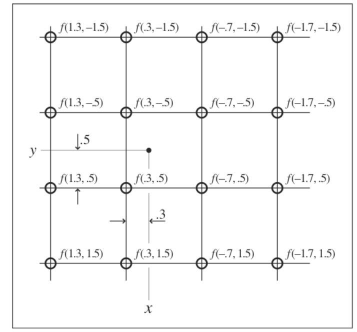
looks like discrete filter,
but offsets are not integers
and filter is continuous

remember placement of filter
relative to grid is variable



Cont.–disc. convolution in 2D

$$(a \star f)(x, y) = \sum_{i,j} a[i, j] f(x - i, y - j)$$



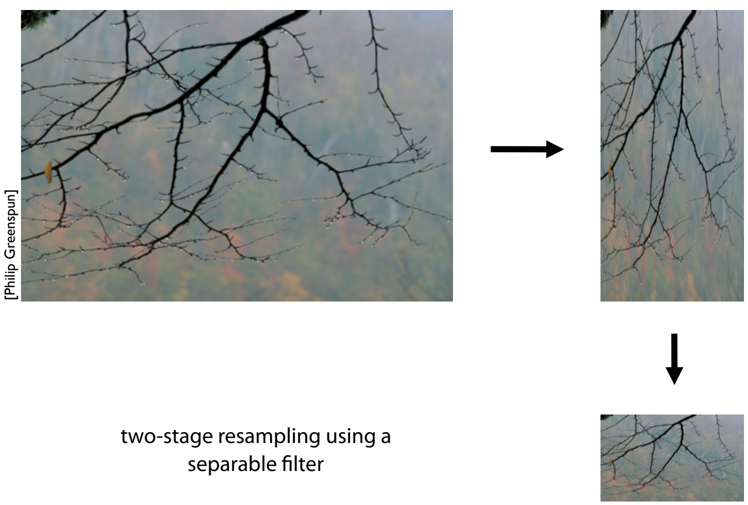
Separable filters for resampling

- just as in filtering, separable filters are useful
  - separability in this context is a statement about a continuous filter, rather than a discrete one:

$$f_2(x, y) = f_1(x) f_1(y)$$

- resample in two passes, one resampling each row and one resampling each column
- intermediate storage required: product of one dimension of src. and the other dimension of dest.
- same yucky details about boundary conditions

[Philip Greenspun]



two-stage resampling using a separable filter

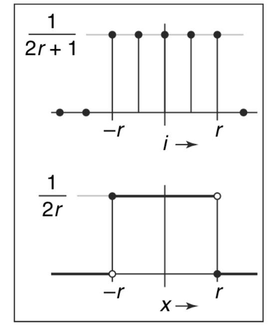
A gallery of filters

- Box filter
  - Simple and cheap
- Tent filter
  - Linear interpolation
- Gaussian filter
  - Very smooth antialiasing filter
- B-spline cubic
  - Very smooth
- Catmull-rom cubic
  - Interpolating
- Mitchell-Netravali cubic
  - Good for image upsampling

Box filter

$$a_{\mathrm{box},r}[i] = \begin{cases} 1/(2r+1) & |i| \le r, \\ 0 & \text{otherwise.} \end{cases}$$

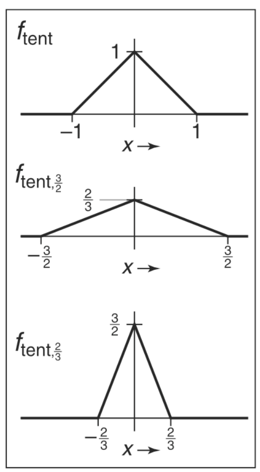
$$f_{\mathrm{box},r}(x) = \begin{cases} 1/(2r) & -r \le x < r, \\ 0 & \text{otherwise.} \end{cases}$$



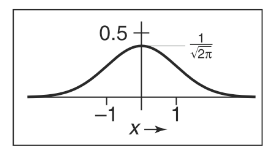
Tent filter

$$f_{\mathrm{tent}}(x) = \begin{cases} 1 - |x| & |x| < 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$f_{\mathrm{tent},r}(x) = \frac{f_{\mathrm{tent}}(x/r)}{r}.$$

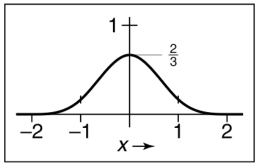


Gaussian filter



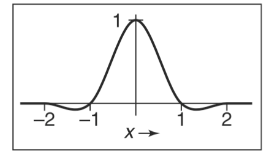
$$f_g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

B-Spline cubic



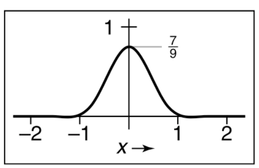
$$f_B(x) = \frac{1}{6}\begin{cases} -3(1-|x|)^3 + 3(1-|x|)^2 + 3(1-|x|) + 1 & -1 \le x \le 1, \\ (2-|x|)^3 & 1 \le |x| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Catmull-Rom cubic



$$f_C(x) = \frac{1}{2}\begin{cases} -3(1-|x|)^3 + 4(1-|x|)^2 + (1-|x|) & -1 \le x \le 1, \\ (2-|x|)^3 - (2-|x|)^2 & 1 \le |x| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Michell-Netravali cubic



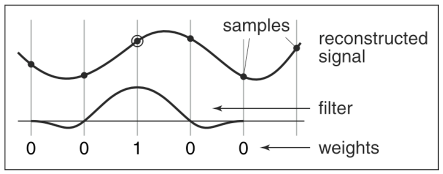
$$\begin{aligned} f_M(x) &= \frac{1}{3}f_B(x) + \frac{2}{3}f_C(x) \\ &= \frac{1}{18}\begin{cases} -21(1-|x|)^3 + 27(1-|x|)^2 + 9(1-|x|) + 1 & -1 \le x \le 1, \\ 7(2-|x|)^3 - 6(2-|x|)^2 & 1 \le |x| \le 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Effects of reconstruction filters

- For some filters, the reconstruction process winds up implementing a simple algorithm
- Box filter (radius 0.5): nearest neighbor sampling
  - box always catches exactly one input point
  - it is the input point nearest the output point
  - so output[*i*, *j*] = input[round(*x*(*i*)), round(*y*(*j*))]
    - *x*(*i*) computes the position of the output coordinate *i* on the input grid
- Tent filter (radius 1): linear interpolation
  - tent catches exactly 2 input points
  - weights are *a* and (1 – *a*)
  - result is straight-line interpolation from one point to the next

Properties of filters

- Degree of continuity
- Impulse response
- Interpolating or no
- Ringing, or overshoot

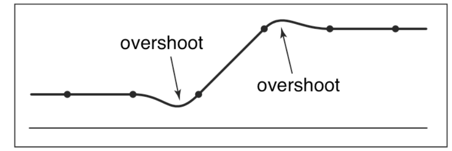


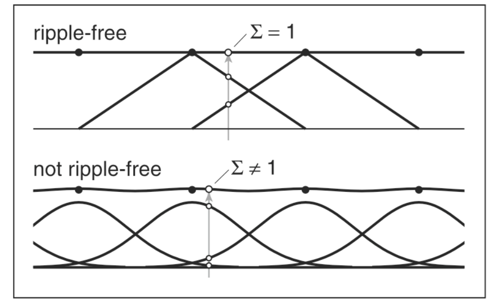
interpolating filter used for reconstruction

Ringing, overshoot, ripples

- Overshoot
  - caused by negative filter values
- Ripples
  - constant in, non-const. out
  - ripple free when:

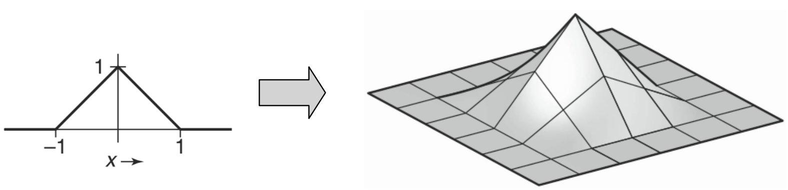
$$\sum_i f(x+i) = 1 \quad \text{for all } x.$$

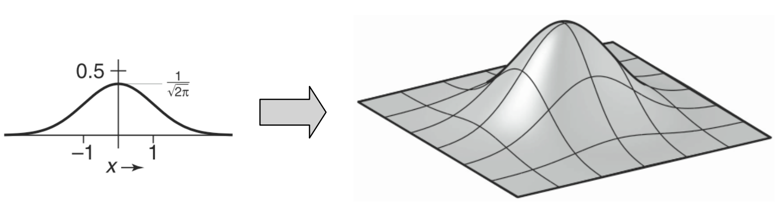




Constructing 2D filters

- Separable filters (most common approach)





Yucky details

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - clip filter (black)
    - wrap around
    - copy edge
    - reflect across edge
    - vary filter near edge

[Philip Greenspun]

Yucky details

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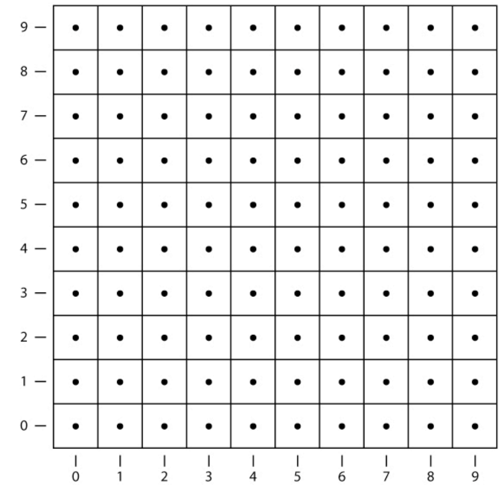
[Philip Greenspun]

Reducing and enlarging

- Very common operation
  - devices have differing resolutions
  - applications have different memory/quality tradeoffs
- Also very commonly done poorly
- Simple approach: drop/replicate pixels
- Correct approach: use resampling

Resampling example



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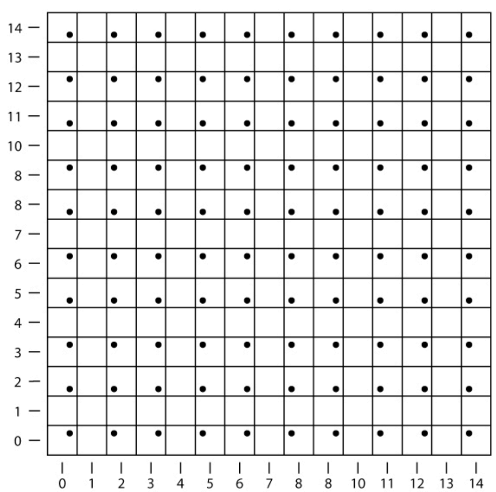

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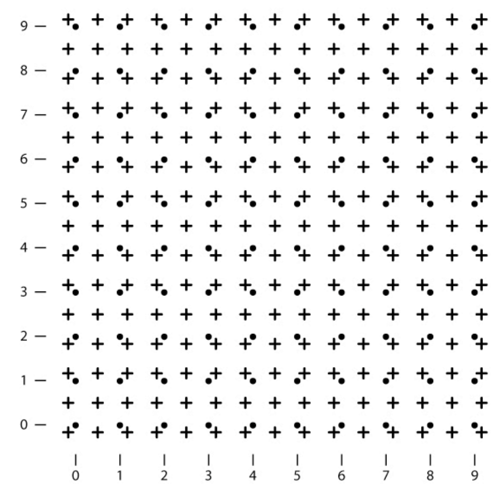

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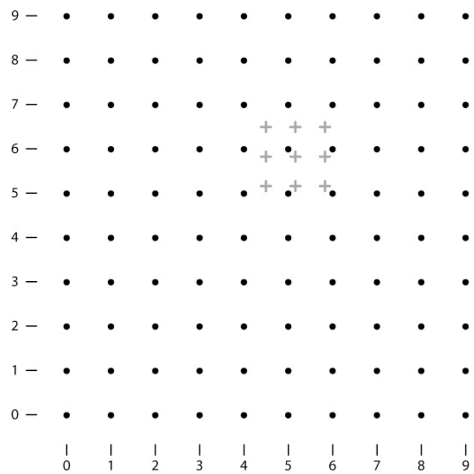
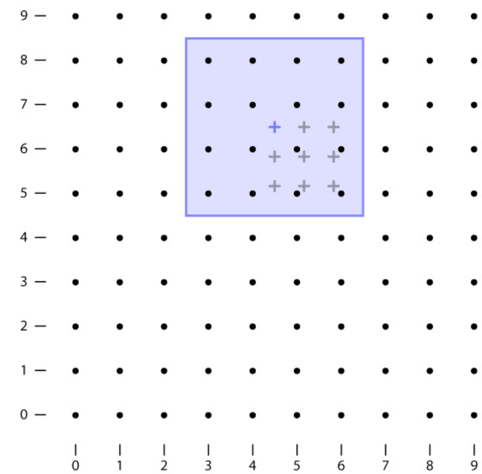
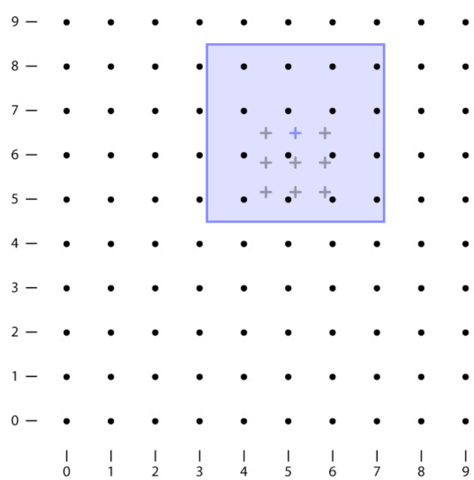


Resampling example



Resampling example



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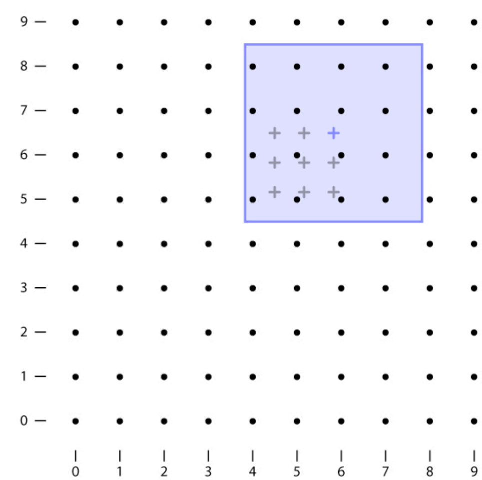

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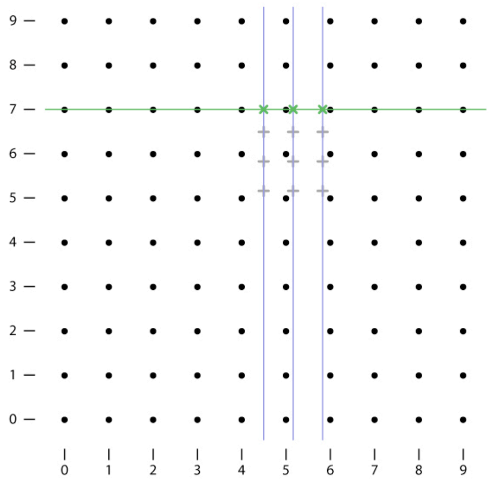

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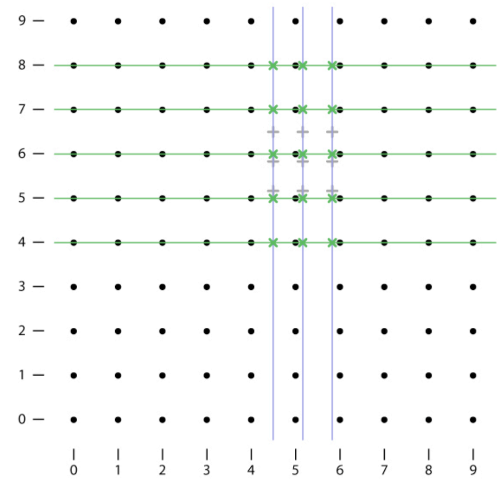

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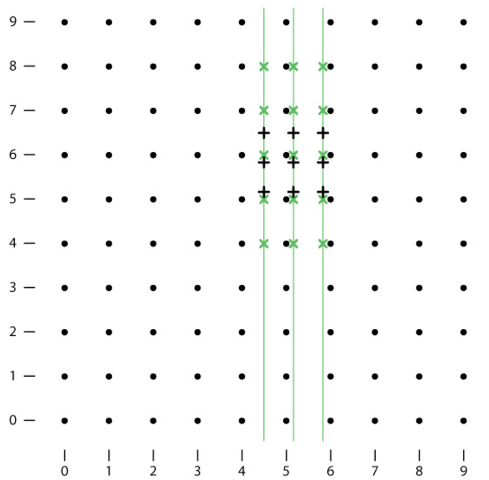

Reducing and enlarging

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  - devices have differing resolutions
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- Also very commonly done poorly
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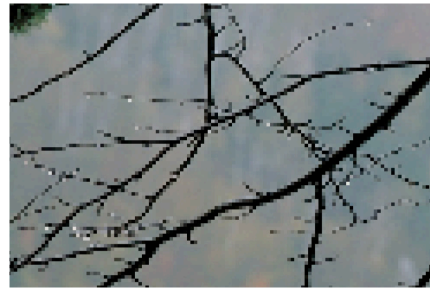
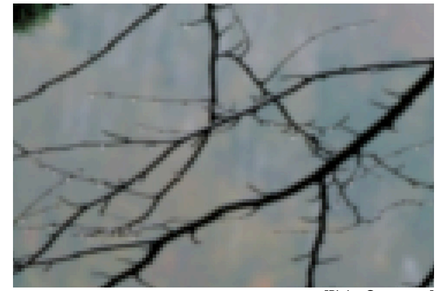
- Correct approach: use resampling

1000 pixel width

[Philip Greenspun]



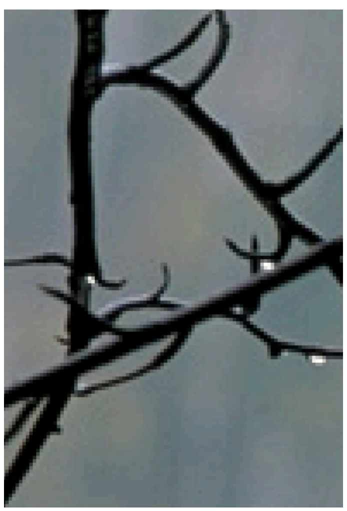


[Philip Greenspun]

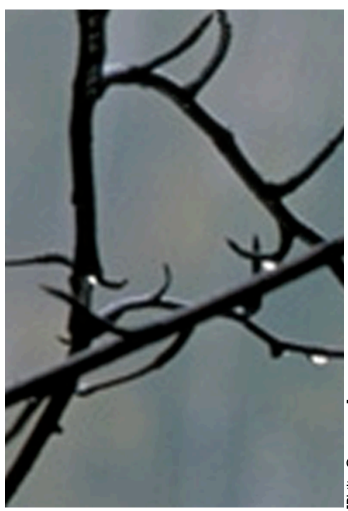
by dropping pixels

gaussian filter

250 pixel width



box reconstruction filter



bicubic reconstruction filter

[Philip Greenspun]

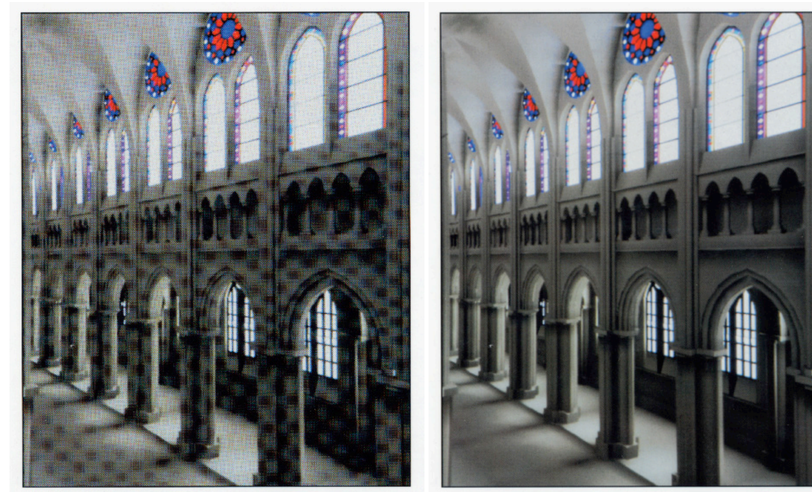
4000 pixel width

Types of artifacts

- Garden variety
  - what we saw in this natural image
  - fine features become jagged or sparkle
- Moiré patterns

[Hearn & Baker cover]

600ppi scan of a color halftone image



by dropping pixels gaussian filter

[Hearn & Baker cover]

downsampling a high resolution scan

Types of artifacts

- Garden variety
  - what we saw in this natural image
  - fine features become jagged or sparkle
- Moiré patterns
  - caused by repetitive patterns in input
  - produce large-scale artifacts; highly visible
- These artifacts are *aliasing* just like in the audio example earlier
- How do I know what filter is best at preventing aliasing?
  - practical answer: experience
  - theoretical answer: there is another layer of cool math behind all this
    - based on Fourier transforms
    - provides much insight into aliasing, filtering, sampling, and reconstruction