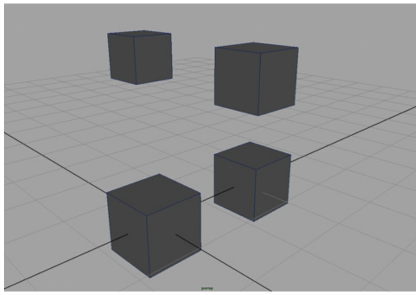
3D Transformations

CS 4620 Lecture 3

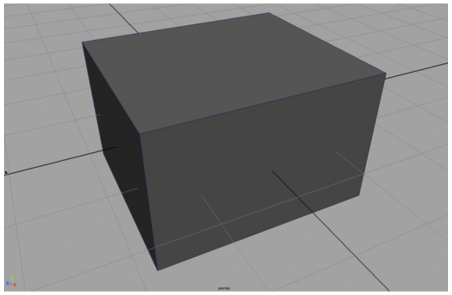
Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



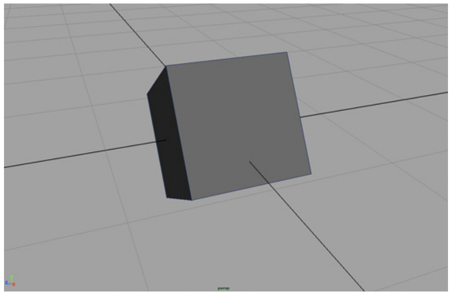
Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



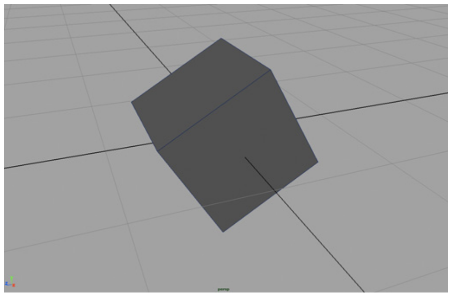
Rotation about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



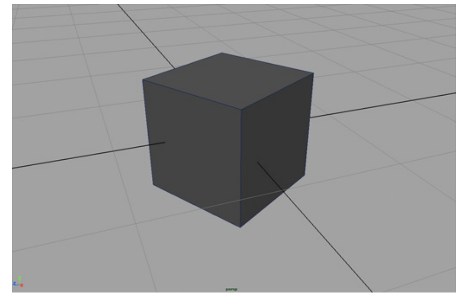
Rotation about *x* axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Rotation about y axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Transformations in OpenGL

- Stack-based manipulation of model-view transformation, M
- **`glMatrixMode(GL_MODELVIEW)`** Specifies model-view matrix
- **`glLoadIdentity()`** M ← 4x4 identity
- **`glTranslatef(float ux, float uy, float uz)`** M ← M T
- **`glRotatef(float theta, float ux, float uy, float uz)`** M ← M R
- **`glScalef(float sx, float sy, float sz)`** M ← M S
- **`glLoadMatrixf(float[] A)`** M ← A (Note: column major)
- **`glMultMatrixf(float[] A)`** M ← M A (Note: column major)
- Manipulate matrix stack using:
  - **`glPushMatrix()`**
  - **`glPopMatrix()`**

Transformations in OpenGL

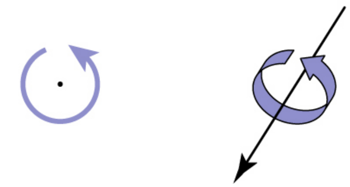
```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
{// Draw something:
   glPushMatrix();
   glTranslatef(...);
   glRotatef(15f, ...);
   {// set color and draw simplices
      glBegin(GL_TRIANGLES);
      glColor3f(...);
      glVertex3f(...);
      glVertex3f(...);
      glVertex3f(...);
      glEnd();
   }
   glPopMatrix(); // toss old transform
 }
{// Draw something else:
   glPushMatrix();
   ...
   glPopMatrix(); // toss old transform
 }
```


Transformations in OpenGL

– **Tutors demo**

General Rotation Matrices

- A rotation in 2D is around a point
- A rotation in 3D is around an axis
  - so 3D rotation is w.r.t a line, not just a point
  - there are many more 3D rotations than 2D
    - a 3D space around a given point, not just 1D



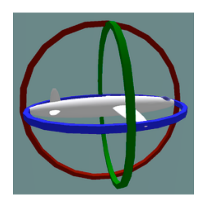
2D 3D

Properties of Rotation Matrices

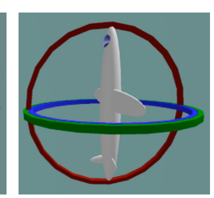
- Columns of R are mutually orthonormal: $RR^T=R^TR=I$
- Right-handed coordinate systems: $\det(R)=1$
  - Recall definition of $\det(R)=r_1^T(r_2 x r_3)$
- Such 3x3 rotation matrices belong to group, SO(3)
  - Special orthogonal
  - Special --> det(R)=1

Specifying rotations

- In 2D, a rotation just has an angle
  - if it's about a particular center, it's a point and angle
- In 3D, specifying a rotation is more complex
  - basic rotation about origin: unit vector (axis) and angle
    - convention: positive rotation is CCW when vector is pointing at you
  - about different center: point (center), unit vector, and angle
    - this is redundant: think of a second point on the same axis...
- Alternative: Euler angles
  - stack up three coord axis rotations
    - ZYX case: Rz(az)*Ry(ay)*Rx(ax)
  - degeneracies exist for some angles
  - E.g., gimbal lock
  - Black board



Unlocked



Gimbal lock

Coming up with the matrix

- Showed matrices for coordinate axis rotations
  - but what if we want rotation about some random axis?
- Can compute by composing elementary transforms
  - transform rotation axis to align with *x* axis
  - apply rotation
  - inverse transform back into position
- Just as in 2D this can be interpreted as a similarity transform

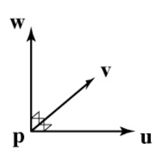
Building general rotations

- Using elementary transforms you need three
  - translate axis to pass through origin
  - rotate about *y* to get into *x*-*y* plane
  - rotate about *z* to align with *x* axis
- Alternative: construct frame and change coordinates
  - choose *p*, *u*, *v*, *w* to be orthonormal frame with *p* and *u* matching the rotation axis
  - apply similarity transform $T = F\,R_x(\theta)\,F^{-1}$

Orthonormal frames in 3D

- Useful tools for constructing transformations
- Recall rigid motions
  - affine transforms with pure rotation
  - columns (and rows) form right-handed ONB
    - that is, an **o**rtho**n**ormal **b**asis

$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Building 3D frames

- Given a vector **a** and a secondary vector **b**
  - The **u** axis should be parallel to **a**; the **u**–**v** plane should contain **b**
    - **u** = **u** / ||**u**||
    - **w** = **u** x **b**; **w** = **w** / ||w||
    - **v** = **w** x **u**
- Given just a vector **a**
  - The **u** axis should be parallel to **a**; don't care about orientation about that axis
    - Same process but choose arbitrary **b** first
    - Good choice is not near **a**: e.g. set smallest entry to 1

Building general rotations

- Alternative: construct frame and change coordinates
  - choose *p*, *u*, *v*, *w* to be orthonormal frame with *p* and *u* matching the rotation axis
  - apply similarity transform $T = F R_x(\theta) F^{-1}$
  - interpretation: move to *x* axis, rotate, move back
  - interpretation: rewrite *u*-axis rotation in new coordinates
  - (each is equally valid)
- Or just derive the formula once, and reuse it (more later)

Derivation of General Rotation Matrix

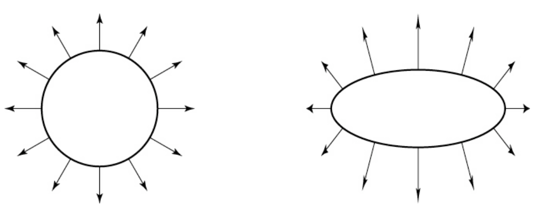
- General 3x3 3D rotation matrix
- General 4x4 rotation about an arbitrary point

Building transforms from points

- Recall: 2D affine transformation has 6 degrees of freedom (DOFs)
  - this is the number of "knobs" we have to set to define one
- Therefore 6 constraints suffice to define the transformation
  - handy kind of constraint: point **p** maps to point **q** (2 constraints at once)
  - three point constraints add up to constrain all 6 DOFs (i.e. can map any triangle to any other triangle)
- 3D affine transformation has 12 degrees of freedom
  - count them by looking at the matrix entries we're allowed to change
- Therefore 12 constraints suffice to define the transformation
  - in 3D, this is 4 point constraints (i.e. can map any tetrahedron to any other tetrahedron)

Transforming normal vectors

- Transforming surface normals
  - differences of points (and therefore tangents) transform OK
  - normals do not --> use inverse transpose matrix



have: $\mathbf{t} \cdot \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$
want: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T X \mathbf{n} = 0$
so set $X = (M^T)^{-1}$
then: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T (M^T)^{-1} \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$