This exam had 11 questions and 120 points total. Students had 120 minutes to complete all the questions. This exam was closed-book and closed-note, but students could consult a prepared sheet of notes (one page, 8 1/2” × 11” written on both sides).

1. [5 points] Given two vectors \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \) such that \( \mathbf{v} \neq \mathbf{0} \), how many flops (accurate to the leading term) are required to compute \( (I - \mathbf{v}\mathbf{v}^T/\langle \mathbf{v}^T\mathbf{v} \rangle)\mathbf{w} \)?

2. [5 points] Same \( \mathbf{v} \) as the previous question, and let \( \mathbf{W} \) be an \( n \times n \) matrix. How many flops (accurate to the leading term) are required to compute \( (I - \mathbf{v}\mathbf{v}^T/\langle \mathbf{v}^T\mathbf{v} \rangle)\mathbf{W} \)?

3. [5 points] Let \( \mathbf{A} \) be an \( m \times n \) matrix with \( m \geq n \). Is it possible to determine whether \( \text{rank}(\mathbf{A}) < n \) with Householder’s QR factorization? If yes, briefly explain how. If no, briefly explain why not.

4. [5 points] The term “Fox Prize” could be a reference to the Leslie Fox Prize in Numerical Analysis. “Fox Prize” might also refer to the prize given to the best singer competing on the Fox TV show American Idol. Which Fox Prize would you rather win? [Note: +1 extra credit if you’ve already won either of these prizes. +2 if you’ve won both.]

5. [10 points] Let \( \mathbf{A} \) be an \( n \times n \) matrix, and let \( \mathbf{B} \) be obtained from \( \mathbf{A} \) by interchanging its first two rows. Explain why \( \mathbf{A} \) and \( \mathbf{B} \) have the same singular values.

6. [10 points] The function \( f : \mathbb{R}^n \to \mathbb{R} \) given by \( f(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|_\infty \), where \( \mathbf{A} \) is an \( m \times n \) matrix and \( \mathbf{b} \) is an \( n \)-vector, is an example of a piecewise linear function. Such a function has the property that there is a partition of \( \mathbb{R}^n \) into a finite number of subsets \( P_1, \ldots, P_s \) such that \( f(\mathbf{x}) \) is affine linear (i.e., has the form \( \mathbf{a}^T\mathbf{x} + b \) for some \( \mathbf{a} \) and \( b \)) on each subset. An even simpler example of a piecewise linear function is \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = |x| \). Explain why Newton’s method for unconstrained minimization is useless for this class of objective functions.

7. [10 points] Let \( \mathbf{v} \in \mathbb{R}^n \) be nonzero. The matrix \( \mathbf{P} = I - \mathbf{v}\mathbf{v}^T/(\langle \mathbf{v}^T\mathbf{v} \rangle) \) is an example of an “orthogonal projector”. Show that it is symmetric and that its eigenvalues are exactly 1, 1, \ldots, 1, 0. [Hint: extend \( \mathbf{v}/\|\mathbf{v}\|_2 \) to an orthogonal basis. These are the eigenvectors.]

8. [20 points] Consider the finite difference method for IVP’s given by

\[
y_{k+1} = \alpha y_k + \beta y_{k-1} + \gamma hf(y_{k+1}, t_{k+1}).
\]

(a) Is this method implicit or explicit?

(b) Determine values of \( \alpha \), \( \beta \) and \( \gamma \) in order to make this method second order. Assume constant stepsize \( h \).
9. [15 points] Let $C$ be a given $n \times n$ matrix and let $A$ be a given $n \times 2$ matrix. Consider the problem of finding $X \in \mathbb{R}^{2 \times n}$ such that $\|AX - C\|_F$ is minimized. Show that this problem can be rewritten as linear least squares for the entries of $X$ by exhibiting the coefficient matrix and right-hand side of a standard-form linear least-squares problem. (Recall that $\| \cdot \|_F$ denotes the Frobenius norm, that is, the square root of sum of squares of matrix entries.)

10. [20 points] Consider the problem of diagonalizing a $2 \times 2$ symmetric matrix $A = [a, b; b, d]$. This can be posed as finding a $2 \times 2$ orthogonal matrix of the form $Q = [c, s; -s, c]$ such that $c^2 + s^2 = 1$ such that the $(2, 1)$ entry of $Q^T AQ$ is zero.
(a) If the $(2, 1)$ entry of $Q^T AQ$ is zero, then $Q^T AQ$ is necessarily diagonal. Why?
(b) Write down a system of two independent nonlinear equations that $(c, s)$ must satisfy in order for $Q$ to diagonalize $A$, and then write down a Newton method to solve these equations (in particular, be sure to write down the Jacobian of those equations). The equations will probably involve $a, b, d$ in the coefficients.

11. [15 points] Consider the initial value problem $\frac{dy}{dt} = f(y, t)$, $y(0) = y_0$, where $y(t) \in \mathbb{R}^n$. Suppose there exists a nonzero $v \in \mathbb{R}^n$ such that $v^T f(y, t) = 0$ for all $y$ and all $t$.
(a) Show that for the true solution $y(t)$ to the IVP, $v^T y(t)$ does not depend on $t$.
(b) Show that for either the backward and forward Euler methods applied to this problem, $v^T y_k$ does not depend on $k$. 