

CS 421: Numerical Analysis
Fall 2004
Prelim 2

Handed out: Friday., Nov. 5.

This exam has four questions. You have 72 hours to answer all questions. The questions are weighted equally even though they are not equally difficult. The exam counts for 20% of your final course grade (same as Prelim 1). This exam is picked up 2:15, Fri., Nov. 5 and due back 2:15, Mon., Nov. 8. Please pick up the exam in lecture and return your solutions in lecture.

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than Heath and the lecture notes, then you must cite your sources.

You may also consult the web and other on-line resources. But you may not make any posting or send any email concerning the exam questions.

Academic integrity. You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until the afternoon of Tues., Nov. 9 because some students may be handing in the exam late. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else's lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: "I have neither given nor received unpermitted assistance on this exam."

Help from the instructor. The only help available will be clarification of the questions. No help will be given towards finding a solution.

Late acceptance policy. Solutions turned in up to 24 hours late will be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are returned on time. No solutions will be accepted more than 24 hours late.

1. Given a matrix $Q_1 \in \mathbf{R}^{n \times k}$ with orthonormal columns, show how to compute an orthogonal matrix $Q \in \mathbf{R}^{n \times n}$ such that $Q(:, 1 : k) = Q_1$. The input matrix Q_1 is presented explicitly, but matrix Q may be represented in implicit form, e.g., as a product of Householder transformations.

Analyze the number of flops required by your algorithm, accurate to the leading terms in n and k .

[Hint: Carry out QR factorization of Q_1 . Make sure the signs come out correctly.]

2. Show that there is a hidden power method in the shifted QR iteration as follows. Let $U^{(k)} = Q^{(0)} \dots Q^{(k-1)}$ as in lecture. Show that there is an inverse shifted power method involving the matrix A^T and the last column of $U^{(k)}$. (Hint for this part: In lecture, an equation was presented to demonstrate the presence of a power method in the shifted QR iteration. Apply the $-T$ operator, that is, inverse transpose, to that equation.)

Then explain that this makes sense because the last column of Q , where $A = QTQ^T$ (Schur factorization), is an eigenvector of A^T . (Why?)

3. Consider two affine subspaces H and J presented in implicit form:

$$H = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$$

and

$$J = \{\mathbf{x} : B\mathbf{x} = \mathbf{c}\}$$

where A is a given $k \times n$ matrix of rank k , \mathbf{b} is a given k -vector, B is a given $l \times n$ matrix of rank l , and \mathbf{c} is a given l -vector. Assume $k + l \leq n$. Propose an algorithm that computes an implicit representation for $H \cap J$, which is also an affine space. In the usual case, $H \cap J$ will be of dimension $n - k - l$. Your algorithm should detect if the dimension is different from $n - k - l$. (If it is different, then your algorithm can simply terminate upon this determination without finding $H \cap J$.) Your algorithm should return an implicit representation in which the defining matrix, let's call it C , is $(k + l) \times n$ and in which C has orthonormal rows.

[Hint: Apply the SVD to $[A; B]$.]

4. (a) Redo question 3 of PS4 in the degenerate case that $\mathbf{u}^T \mathbf{v} = 0$. In particular, determine the n eigenvalues of $A = \mathbf{u}\mathbf{v}^T$ in this case. Explain whether A is diagonalizable. [Hint: Consider an orthogonal matrix Q whose first column is a multiple of \mathbf{u} and whose second is a multiple of \mathbf{v} . Indeed, such a matrix exists by Q1. Show $\mathbf{u}\mathbf{v}^T$ can be reduced to upper triangular form (i.e., Schur factorization) using this orthogonal matrix Q . Could an upper triangular matrix all of whose diagonal entries are equal be diagonalizable?]
- (b) What does the power method do when applied to an upper triangular matrix A all of whose diagonal entries are zero?