CS 421: Numerical Analysis
Fall 2002
Problem Set 4

Handed out: Fri., Oct. 25.

Due: Mon., Nov. 4 in lecture.

1. (a) Derive an algorithm to solve the following problem. Given $A \in \mathbf{R}^{m\times n}$ find $\mathbf{x} \in \mathbf{R}^m$ and $\mathbf{y} \in \mathbf{R}^n$ to maximize $\mathbf{x}^T A\mathbf{y}$, subject to the constraints that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$. In fact, show also that the maximum value of $\mathbf{x}^T A\mathbf{y}$ under these constraints is equal to $\|A\|_2$.

   [Hint for the algorithm: use the SVD. Hint for showing that $\|A\|_2 = \sigma_1(A)$ is the maximum possible value: use the Cauchy-Schwarz inequality.]

   (b) The following problem arises in structural mechanics. Let $A \in \mathbf{R}^{n\times n}$ be symmetric. The problem is to find $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ that maximize $\mathbf{x}^T A\mathbf{y}$ subject to the constraints that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$, and $\mathbf{x}^T\mathbf{y} = 0$. Develop an algorithm to solve this problem, and justify the steps of the algorithm.

   [Hints: (1) First solve the problem in the very special case that A is 2×2 of the form $A = \text{diag}(a, -a)$. You can use part (a) of this question to argue that the solution you found is optimal. (2) Then generalize to the case of an $n \times n$ diagonal matrix whose maximum diagonal entry is a and whose minimum diagonal entry is $-a$. (3) Then generalize to an arbitrary $n \times n$ diagonal matrix by claiming that the optimal $\mathbf{x}, \mathbf{y}$ are unchanged if A is replaced by $A + \alpha I$. (4) Finally, use diagonalization and change of variables to solve for the general case.]

2. Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be a list of vectors in $\mathbf{R}^n$. The problem is to determine whether they all lie in an affine set (defined in PS3), and if so, what is the dimension of the smallest affine set that contains them. Propose an algorithm for this problem based on the SVD that finds the affine set in implicit form. The algorithm should involve a tolerance so that it still works even if the vectors have errors on the order of machine-epsilon in their entries. Hint: suppose $\mathbf{v}_1, \ldots, \mathbf{v}_p$ all lie in an affine set H of dimension k. What can be said about the rank of the matrix $[\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \ldots, \mathbf{v}_p - \mathbf{v}_1]$?

3. (a) Show that the eigenvalues of an upper triangular matrix $U \in \mathbf{R}^{n\times n}$ are exactly the diagonal entries of U. [Hint: Which upper triangular matrices are singular?]

   (b) Propose an algorithm that takes as input an upper triangular matrix $U \in \mathbf{R}^{n\times n}$ with distinct diagonal entries and an index $i \in \{1, \ldots, n\}$ and computes an eigenvector $\mathbf{x}$ such that $U\mathbf{x} = U(i, i)\mathbf{x}$. [Hint: try back substitution.]

4. Download some black-and-white photos from the web. (Or you can download color photos and convert them to black-and-white by adding together the r,g,b values.) Load them in Matlab as in PS3 using `imread`. Note that a rectangular black-and-white image

may be regarded as a matrix, where entries indicate gray-level values. Approximate this matrix using $U(:, 1 : r)\Sigma(1 : r, 1 : r)V(:, 1 : r)^T$ based on the theorem in class starting from the SVD. The `svd` function in Matlab computes the SVD. Display the resulting approximation on your paper for several images and for several different values of r, including values as small as $r = 3$. Comment on your results. How small can r be before the image becomes unrecognizable? Note that image compression routines used in practice (like GIF and JPEG) tend to work better than this SVD-based approach since the SVD-based compression algorithm was not specifically designed for images.

Hand in listings of all m-files, the desired printouts of images and a paragraph of concluding results.