CS 421: Numerical Analysis
Fall 2002
Prelim 1

Handed out: Thurs., Sep. 26.

This test has five questions and lasts 75 minutes. All the questions are weighted equally even though they are not equally difficult. Write your answers in the exam booklet. This test is closed-book and closed-note, but you may consult a 8.5-by-11 sheet of paper that you have prepared in advance.

1. Let $\mathbf{x}$ be a vector in $\mathbf{R}^n$. (a) Show that
$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty.$$
(b) Exhibit two nonzero vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^n$ such that the first inequality of part (a) is tight (i.e., is satisfied as an equation) for $\mathbf{x}_1$, while the second inequality is tight for $\mathbf{x}_2$.

2. Consider the function $f(x) = \cos x - 1$. (a) Show that the obvious way for evaluating this function is prone to catastrophic cancellation for x close to 0. (b) Propose an alternative way to evaluate this function when x is close to 0. [Hint for (b): recall $\cos 2\theta = \cos^2\theta - \sin^2\theta$.]

3. *Threshold pivoting* is a strategy sometimes used in place of partial pivoting within Gaussian elimination applied to an $n \times n$ matrix A. In threshold pivoting, any uneliminated entry $A(p,k)$ in the pivot column k may be selected as pivot provided $|A(p,k)| \geq \alpha \max|(A(k:n,k))|$ where α is a parameter between 0 and 1. (For example, $\alpha = 1$ would be partial pivoting.) Assuming threshold pivoting is used, derive an upper bound on $\|L\|_\infty$ in terms of n and α, where L is the lower triangular factor resulting from elimination.

4. Consider the problem of evaluating a real-valued differentiable function $f(x)$ of a scalar variable x. The condition number of this problem for argument x_1 is sometimes defined to be $|f'(x_1) \cdot x_1|/|f(x_1)|$. Explain why. [Hint: consider small relative perturbations to x_1. The derivative comes from a Taylor approximation.]

5. Suppose Gaussian elimination with pivoting is performed on a nonsingular $2n \times 2n$ matrix with block structure
$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$
where all the blocks are size $n \times n$. Show that both factors L and U will have a block of zero entries, and determine the number of flops (accurate to the leading term) for computing the $P^T LU$ factorization of this matrix.