CS 421: Numerical Analysis
Fall 2000
Problem Set 2


Due: Fri., Sep. 29 in lecture.

1. Exercise proposed by T. Coleman: (a) Show that for any \( A \in \mathbb{R}^{n \times n}, \| A \|_2 \leq \| A \|_F. \)
   (b) Find a \( 2 \times 2 \) diagonal matrix \( A \) such that \( \| A \|_2 = \| A \|_F. \)

2. Let \( A \) be a symmetric positive semidefinite matrix.
   (a) Show that \( A(1, 1) \) must be nonnegative.
   (b) Show that if \( A(1, 1) = 0 \), then the whole first row and column of \( A \) must be all zeros.
   These two facts play a role in an efficient algorithm for testing whether a matrix is positive semidefinite.

3. Let \( U \) be an \( n \times n \) nonsingular upper triangular matrix. (a) Show that \( \| U^{-1} \|_\infty \geq 1/\min_i |U(i, i)|. \) This fact leads to a simple but not very reliable condition-number estimator (namely, \( \| U^{-1} \|_\infty \approx 1/\min_i |U(i, i)| \)) for upper triangular matrices. (b) In fact, show that this estimator is not reliable by constructing a \( 2 \times 2 \) upper triangular matrix \( U \) in which \( \| U^{-1} \|_\infty \geq 10^8/\min_i |U(i, i)|. \)

4. This question requires Matlab programming. Consider two different ways to generate an \( n \times n \) unit lower triangular matrix \( L \) all of whose entries are at most 1 in magnitude. Method 1 is to generate the matrix directly by putting random numbers chosen from the interval \([-1, 1]\) below the diagonal (in Matlab, you need the \texttt{rand} function, the \texttt{triu} function, and the \texttt{eye} function). Method 2 is to generate a square matrix \( A \) at random, compute its \( P^T L U \) factorization (in Matlab, use the \texttt{lu} function), and then save \( L \) (ignore \( P \) and \( U \)).
   For each of these two methods, generate matrices of varying sizes up to \( n = 200. \) For each \( L, \) compute the \( \infty \)-norm of \( L^{-1}. \) Make two plots: one showing \( \| L^{-1} \|_\infty \) versus \( n \) for Method 1, and the other for Method 2. The two plots should behave quite differently, and the reason for this difference is not completely understood.
   Hand in: listings of m-files, sample runs, two plots, and a paragraph of conclusions.