This exam had 13 questions and 120 points total. The students had 120 minutes to complete all the questions. This exam was closed-book and closed-note, but students could consult a prepared sheet of notes (one page, \(8\frac{1}{2} \times 11\) written on both sides).

The following questions are short answer—a single phrase or formula suffices.

1. [5 points] Give a reason why Cholesky factorization is preferable to Gaussian elimination with partial pivoting (GEPP) for solving symmetric positive definite linear systems.

2. [5 points] Give an example of an unstable algorithm.

3. [5 points] How many flops (accurate to the leading term) are required to form \(A^T A\), where \(A\) is an \(m \times n\) matrix?

4. [5 points] Newton’s method for solving nonlinear equations doesn’t always converge. Name something that could go wrong with the method to prevent or hinder convergence.

5. [5 points] Consider a finite-difference method for integrating an initial value problem with time-step \(h\). Suppose that on the \(k\)th step, the truncation error introduced on that step (i.e., disregarding errors from previous steps) is of the form \(y'''(t_k)h^3/12\). What is the order of this method?

6. [5 points] Write down a \(2 \times 2\) rank-one matrix.

7. [5 points] In the inverse shifted power method, what is a desirable property of the shift in order to attain fast convergence?

8. [5 points] Bill Clinton, the U.S. President, is not a numerical analyst. Can you name any other famous person who is not a numerical analyst?
These questions require longer answers.

9. **[15 points]** Let $A \in \mathbb{R}^{m \times n}$ be factored as $QR$, where $Q \in \mathbb{R}^{n \times n}$ is orthogonal and $R \in \mathbb{R}^{m \times n}$ is upper triangular. Assume $A$ has full rank. Show that $A$ and $Q(:, 1:n)$ have the same range-space, and in fact, come up with an algorithm to solve $Ax = Q(:, 1:n)y$ for $x$ given $y$. Come up with a second algorithm that solves for $y$ given $x$.

[Note: $Q(:, 1:n)$ means the submatrix of $Q$ formed by its first $n$ columns.]

10. **[15 points]** Given $A \in \mathbb{R}^{n \times n}$ and a nonzero vector $y \in \mathbb{R}^n$, suppose the statement

$$y = A \ast y(n:-1:1);$$

is executed repeatedly in Matlab. Clearly this is some kind of a power method. What would you expect it to converge to (after normalization)?

[Note: In case you are not familiar with Matlab notation, the above statement means: set $y$ to the product $A z$, where $z$ is defined as the vector whose entries are the entries in $y$ in reverse order.]

11. **[15 points]** Describe the tradeoffs involved in using an explicit versus implicit method for integrating an initial value problem. Under what circumstances does one or the other have a clear advantage?

12. **[15 points]** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ have the property that, for each $i = 1, \ldots, n$, $f_i$ depends only on $x_i, \ldots, x_n$ (and not on $x_1, \ldots, x_{i-1}$). Show that the Newton step for solving $f(x) = 0$ can be computed especially efficiently. How many flops (accurate to the leading term) are required to compute the Newton step (once the Jacobian is known)?

13. **[20 points]** Consider the **implicit midpoint (IM) rule** for integrating an ODE. This rule is defined by the formula:

$$y_{k+1} = y_k + h_k f((y_k + y_{k+1})/2, (t_k + t_{k+1})/2).$$

Consider applying this rule to the IVP

$$\frac{dy}{dt} = Ay,$$

$$y(0) = y_0,$$

where $A$ is an $n \times n$ matrix. Show that if all the eigenvalues of $A$ have negative real parts, then $y_k$ computed by IM will tend to zero as $k \to \infty$. Assume a fixed time-step (i.e., $h_k = h$ independent of $k$). [Hint: The matrix $(I + \alpha A)^{-1}(I + \beta A)$ has the same eigenvectors as $A$. Why? How are its eigenvalues related to those of $A$?]