CS 421 – **Problem Set 4**
Due: Wednesday, October 27, 1999

1. An important quantity in the trust region approach, as described in class, is the variable "*ratio*". Show why a negative value of *ratio* implies $f(x_k + s_k) > f(x_k)$ where s_k is the "trial step" computed by solving,

$$\min_s \{ g_k^T s + \frac{1}{2} s^T H_k s : \|s\| \leq \Delta_k \}.$$

2. Assume we are using a line search algorithm to solve a nonlinear minimization problem and suppose that the two line search conditions, given in class, are satisfied at x_{k+1}, where $x_{k+1} = x_k + \alpha_k s_k$. Let B_k be the current (symmetric positive definite) approximation to the Hessian matrix. Show that it is possible to obtain B_{k+1} via the BFGS update.

3. (a) Give a specific 2-dimensional example of a quadratic function $q(x)$, with a positive semi-definite Hessian matrix, such that $q(x)$ is unbounded below.
   (b) Change your example so that $q(x)$ has a positive semi-definite Hessian matrix but $q(x)$ is bounded below and has an infinite number of global minimizers.

4. Recall the derivation of the BFGS update (symmetric positive definite update) given in class. Recall that given the Cholesky factor of the current approximation, $B = LL^T$, the new approximation satisfies

$$B_+ = M_+ M_+^T$$

where $M_+ = L_+ + E$ and $E = \frac{(y - Lw)w^T}{w^T w}$ with $w = \alpha L^T s$ and

$$\alpha^2 = \frac{y^T s}{s^T L L^T s}. \tag{1}$$

Assume $y^T s > 0$. Show that

$$B_+ = B + \frac{yy^T}{s^T y} - \frac{Bss^T B}{s^T B s}$$

regardless of which root for α is taken in (1).

5. The purpose of this question is to introduce you to the idea of a contour map of a function of 2 variables, initially with respect to a quadratic function.

   A quadratic function $q(x) = \frac{1}{2}x^T Hx$ can be generated as follows. First, choose a diagonal matrix. Second, generate a random orthogonal matrix with the Matlab command $Q = orth(rand(2))$. Finally, set $H = Q^T DQ$.

   Create four example quadratics with $D = diag(1, 1)$, $D = diag(10, 1)$, $D = diag(1000, 1)$, $D = diag(2, -10)$, respectively. Plot a contour map for each case (centered at the origin) and comment on your pictures.

6. The symmetric secant update is given by $B_+ = B + C$ where C solves

$$\min\{\|C\|_F;\ Cs = r, C = C^T\} \tag{2}$$

   and $r = y - Bs$. The solution is

$$C = \frac{rs^T + sr^T}{s^T s} - (s^T r)\frac{ss^T}{(s^T s)^2}. \tag{3}$$

   This update can be derived by considering (2) and introducing an orthogonal matrix Q such that $Q^T[s, r]$ is an upper triangular n-by-2 matrix. Complete this derivation.