Lecture 22
Normalization
Type “Completeness”? 

Are all well-behaved programs well-typed?
The simply-typed lambda calculus enjoys a remarkable property:

Every well-typed program terminates.
Simply-Typed Lambda Calculus

Syntax

expressions\n\[ e ::= x | \lambda x : \tau. \ e \ | \ e_1 \ e_2 \ | \ () \]

values\n\[ v ::= \lambda x : \tau. \ e \ | \ () \]

types\n\[ \tau ::= \text{unit} \ | \ \tau_1 \rightarrow \tau_2 \]
Simply-Typed Lambda Calculus

Syntax

expressions

\[ e ::= x \mid \lambda x : \tau. e \mid e_1 e_2 \mid () \]

values

\[ v ::= \lambda x : \tau. e \mid () \]

types

\[ \tau ::= \text{unit} \mid \tau_1 \to \tau_2 \]

Dynamic Semantics

\[ E ::= [\cdot] \mid E e \mid v E \]

\[ e \rightarrow e' \]
\[ \frac{E[e] \rightarrow E[e']} {E[e] \rightarrow E[e']} \]
\[ (\lambda x : \tau. e) v \rightarrow e(\{v/x\}) \]
Simply-Typed Lambda Calculus

Static Semantics

\[ \frac{}{\Gamma \vdash () : \text{unit}} \quad \text{T-UNIT} \]

\[ \frac{}{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \quad \text{T-VAR} \]

\[ \frac{\Gamma \vdash x : \tau}{\Gamma, x : \tau \vdash e : \tau'} \quad \text{T-ABS} \]

\[ \frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'} \quad \text{T-APP} \]
Supporting Lemmas

Lemma (Inversion)

- If $\Gamma \vdash x : \tau$ then $\Gamma(x) = \tau$
- If $\Gamma \vdash \lambda x : \tau_1 . e : \tau$ then $\tau = \tau_1 \rightarrow \tau_2$ and $\Gamma, x : \tau_1 \vdash e : \tau_2$.
- If $\Gamma \vdash e_1 e_2 : \tau$ then $\Gamma \vdash e_1 : \tau' \rightarrow \tau$ and $\Gamma \vdash e_2 : \tau'$.
Supporting Lemmas

Lemma (Inversion)

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• If $\Gamma \vdash e_1 e_2 : \tau$ then $\Gamma \vdash e_1 : \tau' \rightarrow \tau$ and $\Gamma \vdash e_2 : \tau'$.

Lemma (Canonical Forms)

• If $\Gamma \vdash v : \text{unit}$ then $v = ()$
• If $\Gamma \vdash v : \tau_1 \rightarrow \tau_2$ then $v = \lambda x : \tau_1 . e$ and $\Gamma, x : \tau_1 \vdash e : \tau_2$. 
First Attempt

Theorem (Normalization)

\( \vdash e : \tau \text{ then there exists a value } v \text{ such that } e \rightarrow^* v. \)
Idea: define a set with the following properties:

- At base types the set contains all expressions satisfying some property.
- At function types, the set contains all expressions such that the property is preserved whenever we apply the function to an argument of appropriate type that is also in the set.
**Logical Relations**

**Idea:** define a set with the following properties:

- At base types the set contains all expressions satisfying some property.
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In our setting, the property will concern normalization...
Definition (Logical Relation)

- $R_{\text{unit}}(e)$ iff $e : \text{unit}$ and $e$ halts.
- $R_{\tau_1 \rightarrow \tau_2}(e)$ iff $e : \tau_1 \rightarrow \tau_2$ and $e$ halts, and for every $e'$ such that $R_{\tau_1}(e')$ we have $R_{\tau_2}(e \ e')$. 
Supporting Lemmas

Lemma

If $R_\tau(e)$ then $e$ halts.
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Lemma

If $\vdash e : \tau$ and $e \rightarrow e'$ then $R_{\tau}(e)$ iff $R_{\tau}(e')$. 
Supporting Lemmas

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If $\vdash e : \tau$ and $e \rightarrow e'$ then $R_\tau(e)$ iff $R_\tau(e')$.

Lemma (Goal)

If $\vdash e : \tau$ then $R_\tau(e)$
Main Lemma

Lemma (Goal – Strengthened)

If

- \( x_1: \tau_1, \ldots, x_k: \tau_k \vdash e: \tau \),
- \( v_1 \) through \( v_k \) are values such that \( \vdash v_1: \tau_1 \) through \( \vdash v_k: \tau_k \), and
- \( R_{\tau_1}(v_1) \) through \( R_{\tau_k}(v_k) \),

then \( R_{\tau}(e\{v_1/x_1\} \ldots \{v_k/x_k\}) \).