Lecture 18
Evaluation Contexts and Definitional Translation
Review: Call-by-Value

Here are the syntax and CBV semantics of $\lambda$-calculus:

$$
e ::= x \mid \lambda x. e \mid e_1 e_2
\nu ::= \lambda x. e$$

$$
e_1 \to e_1' \quad e \to e'
\frac{}{e_1 e_2 \to e_1' e_2}
\frac{}{\nu e \to \nu e'}$$

$$
(\lambda x. e) \nu \to e\{\nu/x\}^{\beta}
$$

There are two kinds of rules: *congruence rules* that specify evaluation order and *computation rules* that specify the “interesting” reductions.
Evaluation Contexts

Evaluation contexts let us separate out these two kinds of rules.
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An evaluation context $E$ is an expression with a “hole” in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= [\cdot] \mid E e \mid \nu E$$
Evaluation contexts let us separate out these two kinds of rules.

An evaluation context $E$ is an expression with a “hole” in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= \cdot | E \, e | \nu \, E$$

We write $E[e]$ to mean the evaluation context $E$ where the hole has been replaced with the expression $e$. 
Examples

\[ E_1 = [\cdot] (\lambda x. x) \]
\[ E_1[\lambda y. y y] = (\lambda y. y y) \lambda x. x \]
Examples

\[ E_1 = \, [\cdot] \, (\lambda x. \, x) \]
\[ E_1[\lambda y. \, y \, y] = (\lambda y. \, y \, y) \, \lambda x. \, x \]

\[ E_2 = (\lambda z. \, z \, z) \, [\cdot] \]
\[ E_2[\lambda x. \, \lambda y. \, x] = (\lambda z. \, z \, z) \, (\lambda x. \, \lambda y. \, x) \]
Examples

\[ E_1 = [\cdot] (\lambda x. x) \]
\[ E_1[\lambda y. y y] = (\lambda y. y y) \lambda x. x \]

\[ E_2 = (\lambda z. z z) [\cdot] \]
\[ E_2[\lambda x. \lambda y. x] = (\lambda z. z z) (\lambda x. \lambda y. x) \]

\[ E_3 = ([\cdot] \lambda x. x x) ((\lambda y. y) (\lambda y. y)) \]
\[ E_3[\lambda f. \lambda g. f g] = ((\lambda f. \lambda g. f g) \lambda x. x x) ((\lambda y. y) (\lambda y. y)) \]
CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV $\lambda$-calculus with just two rules: one for evaluation contexts, and one for $\beta$-reduction.
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With this syntax:

$$E ::= [\cdot] \mid E \, e \mid v \, E$$

The small-step rules are:

$$e \rightarrow e'$$

$$\frac{}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x. \, e) \, v \rightarrow e\{v/x\}}$$
We can also define the semantics of CBN $\lambda$-calculus with evaluation contexts.
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For call-by-name, the syntax for evaluation contexts is different:

$$E ::= [\cdot] \mid E \ e$$
CBN With Evaluation Contexts

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For call-by-name, the syntax for evaluation contexts is different:

$$E ::= [\cdot] \mid E\,e$$

But the small-step rules are the same:

$$e \to e'$$
$$\frac{}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x. \, e)\,e' \to e\{e' / x\}}^{\beta}$$
We know how to encode Booleans, conditionals, natural numbers, and recursion in $\lambda$-calculus.

Can we define a *real* programming language by translating everything in it into the $\lambda$-calculus?
Definitional Translation

We know how to encode Booleans, conditionals, natural numbers, and recursion in $\lambda$-calculus.

Can we define a real programming language by translating everything in it into the $\lambda$-calculus?

In definitional translation, we define a denotational semantics where the target is a simpler programming language instead of mathematical objects.
Multi-Argument $\lambda$-calculus

Let’s define a version of the $\lambda$-calculus that allows functions to take multiple arguments.

$$e ::= x \mid \lambda x_1, \ldots, x_n. e \mid e_0 e_1 \ldots e_n$$
Multi-Argument $\lambda$-calculus

We can define a CBV operational semantics:

$$E ::= [\cdot] | v_0 \ldots v_{i-1} E e_{i+1} \ldots e_n$$

$$(\lambda x_1, \ldots, x_n. e_0) v_1 \ldots v_n \to e_0\{v_1/x_1\}\{v_2/x_2\} \ldots \{v_n/x_n\} \quad \beta$$

The evaluation contexts ensure that we evaluate multi-argument applications $e_0 e_1 \ldots e_n$ from left to right.
Definitional Translation

The multi-argument $\lambda$-calculus isn’t any more expressive than the pure $\lambda$-calculus.

$T[\[x\]] = x$

$T[\[\lambda x_1, \ldots, x_n. e\]] = \lambda x_1. \ldots. \lambda x_n. T[\[e\]]$

$T[\[e_0 e_1 e_2 \ldots e_n\]] = (\ldots((T[\[e_0\]] T[\[e_1\]]) T[\[e_2\])\ldots T[\[e_n\])}$

This translation curries the multi-argument $\lambda$-calculus.
Definitional Translation

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We can define a translation $T[\cdot]$ that takes an expression in the multi-argument $\lambda$-calculus and returns an equivalent expression in the pure $\lambda$-calculus.
Definitional Translation

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We can define a translation $T[\cdot]$ that takes an expression in the multi-argument $\lambda$-calculus and returns an equivalent expression in the pure $\lambda$-calculus.

\[
T[x] = x \\
T[\lambda x_1, \ldots, x_n. e] = \lambda x_1. \ldots \lambda x_n. T[e] \\
T[e_0 e_1 e_2 \ldots e_n] = (\ldots ((T[e_0] T[e_1]) T[e_2]) \ldots T[e_n])
\]

This translation *curries* the multi-argument $\lambda$-calculus.
Syntax

\[ e ::= x \]

\[ | \lambda x. e \]

\[ | e_1 e_2 \]

\[ | (e_1, e_2) \]

\[ | \#1 e \]

\[ | \#2 e \]

\[ | \text{let } x = e_1 \text{ in } e_2 \]

\[ v ::= \lambda x. e \]

\[ | (v_1, v_2) \]
Evaluation Contexts

\[
E ::= [\cdot] \\
| \ E e \\
| \ v E \\
| (E, e) \\
| (v, E) \\
| \#1 E \\
| \#2 E \\
| \text{let } x = E \text{ in } e_2
\]
Products (Pairs) and Let

Semantics

\[
\begin{align*}
e & \rightarrow e' \\
E[e] & \rightarrow E[e']
\end{align*}
\]

\[
(\lambda x. e) \nu \rightarrow e\{\nu/x\} \quad \beta
\]

\[
\begin{align*}
\#1 (v_1, v_2) & \rightarrow v_1 \\
\#2 (v_1, v_2) & \rightarrow v_2
\end{align*}
\]

\[
\text{let } x = v \text{ in } e \rightarrow e\{v/x\}
\]
Products (Pairs) and Let

Translation

\[ \mathcal{T}[x] = x \]
\[ \mathcal{T}[\lambda x. e] = \lambda x. \mathcal{T}[e] \]
\[ \mathcal{T}[e_1 e_2] = \mathcal{T}[e_1] \cdot \mathcal{T}[e_2] \]
\[ \mathcal{T}[\langle e_1, e_2 \rangle] = (\lambda x. \lambda y. \lambda f. f x y) \cdot \mathcal{T}[e_1] \cdot \mathcal{T}[e_2] \]
\[ \mathcal{T}[\#1 e] = \mathcal{T}[e] \cdot (\lambda x. \lambda y. x) \]
\[ \mathcal{T}[\#2 e] = \mathcal{T}[e] \cdot (\lambda x. \lambda y. y) \]
\[ \mathcal{T}[\text{let } x = e_1 \text{ in } e_2] = (\lambda x. \mathcal{T}[e_2]) \cdot \mathcal{T}[e_1] \]
Laziness

Consider the call-by-name $\lambda$-calculus...

Syntax

\[
\begin{align*}
e & ::= x \\
    & \quad \mid e_1 e_2 \\
    & \quad \mid \lambda x. e \\
\end{align*}
\]

\[
\nu ::= \lambda x. e
\]

Semantics

\[
\frac{e_1 \to e_1'}{e_1 e_2 \to e_1' e_2} \quad \frac{e_1 \to e_1'}{(\lambda x. e_1) e_2 \to e_1\{e_2/x\}}^\beta
\]
Laziness

Translation

\[ T[x] = x (\lambda y. y) \]
\[ T[\lambda x. e] = \lambda x. T[e] \]
\[ T[e_1 e_2] = T[e_1] (\lambda z. T[e_2]) \quad \text{if } z \text{ is not a free variable of } e_2 \]
References

Syntax

\[ e ::= x \]

\[ \mid \lambda x. e \]

\[ \mid e_0 \ e_1 \]

\[ \nu ::= \lambda x. e \]
Syntax

\[ e ::= x \]
\[ \quad | \lambda x. e \]
\[ \quad | e_0 e_1 \]
\[ \quad | \text{ref } e \]

\[ \nu ::= \lambda x. e \]
Syntax

\[
e ::= x \mid \lambda x. e \mid e_0 e_1 \mid \text{ref } e \mid !e
\]

\[
v ::= \lambda x. e
\]
Syntax

\[ e ::= x \]
\[ \quad | \lambda x. e \]
\[ \quad | e_0 e_1 \]
\[ \quad | \text{ref } e \]
\[ \quad | !e \]
\[ \quad | e_1 := e_2 \]

\[ \nu ::= \lambda x. e \]
Syntax

\[ e ::= x \]
\[ \quad | \quad \lambda x. \ e \]
\[ \quad | \quad e_0 \ e_1 \]
\[ \quad | \quad \text{ref} \ e \]
\[ \quad | \quad !e \]
\[ \quad | \quad e_1 ::= e_2 \]
\[ \quad | \quad \ell \]

\[ \nu ::= \lambda x. \ e \]
References

Syntax

\[
e ::= x \\
\quad | \lambda x. e \\
\quad | e_0 e_1 \\
\quad | \text{ref } e \\
\quad | !e \\
\quad | e_1 ::= e_2 \\
\quad | \ell \\
\]

\[
v ::= \lambda x. e \\
\quad | \ell \\
\]
Evaluation Contexts

\[ E ::= [\cdot] \]
\[ \mid E \; e \]
\[ \mid v \; E \]
Evaluation Contexts

\[ E ::= [\cdot] \]

\[ | E e \]

\[ | v E \]

\[ | \text{ref } E \]
Evaluation Contexts

\[ E ::= [\cdot] \]
\[ | E \ e \]
\[ | \nu \ E \]
\[ | \text{ref} \ E \]
\[ | \!E \]
Evaluation Contexts

\[ E ::= [\cdot] \]
\[ \mid E \ e \]
\[ \mid \nu \ E \]
\[ \mid \text{ref} \ E \]
\[ \mid !E \]
\[ \mid E ::= e \]
Evaluation Contexts

\[
E ::= [\cdot] \\
| E e \\
| \nu E \\
| \text{ref } E \\
| !E \\
| E ::= e \\
| \nu ::= E
\]


References

Semantics

\[
\begin{align*}
\langle \sigma, e \rangle & \rightarrow \langle \sigma', e' \rangle \\
\langle \sigma, E[e] \rangle & \rightarrow \langle \sigma', E[e'] \rangle
\end{align*}
\]

\[
\frac{l \notin \text{dom}(\sigma)}{\langle \sigma, \text{ref } v \rangle \rightarrow \langle \sigma[l \mapsto v], l \rangle}
\]

\[
\frac{\sigma(l) = v}{\langle \sigma, !l \rangle \rightarrow \langle \sigma, v \rangle}
\]

\[
\langle \sigma, l := v \rangle \rightarrow \langle \sigma[l \mapsto v], v \rangle
\]

\[
\frac{\langle \sigma, (\lambda x. e) v \rangle \rightarrow \langle \sigma, e[v/x] \rangle}{\beta}
\]
Translation

...left as an exercise to the reader. ;-}
Adequacy

How do we know if a translation is correct?
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How do we know if a translation is correct?

Every target evaluation should represent a source evaluation...

**Definition (Soundness)**

\[ \forall e \in \mathbf{Exp}_{\text{src}}. \text{ if } \mathcal{T}[e] \rightarrow_{\text{trg}}^* v' \text{ then } \exists v. e \rightarrow_{\text{src}}^* v \]

and \( v' \) equivalent to \( v \)
Adequacy

How do we know if a translation is correct?

Every target evaluation should represent a source evaluation...

**Definition (Soundness)**

\[ \forall e \in E^{\text{src}}. \text{if } T[e] \xrightarrow{\text{trg}} v' \text{ then } \exists v. e \xrightarrow{\text{src}} v \]

and \( v' \) equivalent to \( v \)

...and every source evaluation should have a target evaluation:

**Definition (Completeness)**

\[ \forall e \in E^{\text{src}}. \text{if } e \xrightarrow{\text{src}} v \text{ then } \exists v'. T[e] \xrightarrow{\text{trg}} v' \]

and \( v' \) equivalent to \( v \)