Lecture 8
Denotational Semantics Proofs
Kleene Fixed-Point Theorem

Definition (Scott Continuity)

A function $F$ is Scott-continuous if for every chain $X_1 \subseteq X_2 \subseteq \ldots$ we have $F(\bigcup_i X_i) = \bigcup_i F(X_i)$. 
Kleene Fixed-Point Theorem

Definition (Scott Continuity)

A function $F$ is Scott-continuous if for every chain $X_1 \subseteq X_2 \subseteq \ldots$ we have $F(\bigcup_i X_i) = \bigcup_i F(X_i)$.

Theorem (Kleene Fixed Point)

Let $F$ be a Scott-continuous function. The least fixed point of $F$ is $\bigcup_i F^i(\emptyset)$. 
Denotational Semantics for IMP Commands

\[ C[\text{skip}] = \{(\sigma, \sigma)\} \]

\[ C[x := a] = \{(\sigma, \sigma[x \rightarrow n]) \mid (\sigma, n) \in A[a]\} \]

\[ C[c_1; c_2] = \{(\sigma, \sigma') \mid \exists \sigma''. ((\sigma, \sigma'') \in C[c_1] \land (\sigma'', \sigma') \in C[c_2])\} \]

\[ C[\text{if } b \text{ then } c_1 \text{ else } c_2] = \{(\sigma, \sigma') \mid (\sigma, \text{true}) \in B[b] \land (\sigma, \sigma') \in C[c_1]\} \cup \{(\sigma, \sigma') \mid (\sigma, \text{false}) \in B[b] \land (\sigma, \sigma') \in C[c_2]\} \]

\[ C[\text{while } b \text{ do } c] = \text{fix}(f) \]
where \( F(f) = \{(\sigma, \sigma) \mid (\sigma, \text{false}) \in B[b]\} \cup \{(\sigma, \sigma') \mid (\sigma, \text{true}) \in B[b] \land \exists \sigma''. ((\sigma, \sigma'') \in C[c] \land (\sigma'', \sigma') \in f)\} \)
skip; c and c; skip are equivalent.
skip; c and c; skip are equivalent.

$C[\text{while false do } c]$ is equivalent to...
skip; c and c; skip are equivalent.

$C[\text{while false do } c]$ is equivalent to skip.
skip; c and c; skip are equivalent.

$C[\text{while false do } c]$ is equivalent to skip.

$C[\text{while true do skip}] = ?$