Command Equivalence

Intuitively, two commands are equivalent if they produce the same result under any store...

Definition (Equivalence of commands)

Two commands $c$ and $c'$ are equivalent (written $c \sim c'$) if, for any stores $\sigma$ and $\sigma'$, we have

$$\langle \sigma, c \rangle \downarrow \sigma' \iff \langle \sigma, c' \rangle \downarrow \sigma'.$$
For example, we can prove that every while command is equivalent to its “unrolling”:

**Theorem**

For all \( b \in \text{Bexp} \) and \( c \in \text{Com} \),

\[
\text{while } b \text{ do } c \sim \text{if } b \text{ then } (c; \text{while } b \text{ do } c) \text{ else skip}
\]

**Proof.**

We show each implication separately...
IMP Questions

- Q: Can you write a program that doesn’t terminate?

A: While true do skip

Q: Does this mean that IMP is Turing complete?

A: Not quite... we also need to check the language is not finite state... but IMP has real mathematical integers.

Q: What if we replace \texttt{Int} with \texttt{Int64}?

A: Then we would lose Turing completeness.

Q: How much space do we need to represent configurations during execution of an IMP program?

A: Can calculate a fixed bound!
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Theorem

\[ \forall c \in \text{Com}, \sigma, \sigma', \sigma'' \in \text{Store}. \]

\[ \text{if } \langle \sigma, c \rangle \downarrow \sigma' \text{ and } \langle \sigma, c \rangle \downarrow \sigma'' \text{ then } \sigma' = \sigma''. \]
Theorem

\[ \forall c \in \text{Com}, \sigma, \sigma', \sigma'' \in \text{Store}. \]

if \( \langle \sigma, c \rangle \downarrow \sigma' \) and \( \langle \sigma, c \rangle \downarrow \sigma'' \) then \( \sigma' = \sigma'' \).

Proof.

By structural induction on \( c \)...
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Derivations

Write $\mathcal{D} \vdash y$ if the conclusion of derivation $\mathcal{D}$ is $y$. (Read as “$\mathcal{D}$ proves $y$.”)
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Example:

Given the derivation,

$$
\begin{align*}
\langle \sigma, 6 \rangle & \Downarrow 6 \\
\langle \sigma, 7 \rangle & \Downarrow 7 \\
\langle \sigma, 6 \times 7 \rangle & \Downarrow 42 \\
\langle \sigma, i := 6 \times 7 \rangle & \Downarrow \sigma[i \mapsto 42]
\end{align*}
$$

we would write: $\mathcal{D} \models \langle \sigma, i := 42 \rangle \Downarrow \sigma[i \mapsto 42]$
Induction on Derivations

Remember that every “true” fact given by an inductive definition must have a derivation that “proves” that fact.

For many inductive proofs, it’s useful to visualize the derivation tree for each fact.
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For many inductive proofs, it’s useful to visualize the derivation tree for each fact.

In each case in an inductive proof, we assume that the property $P$ holds for the rule’s premises and prove it for the rule’s conclusion.

Those premises each *also* have derivations.

A derivation $\mathcal{D}'$ is an immediate subderivation of $\mathcal{D}$ if $\mathcal{D}' \models z$ where $z$ is one of the premises used of the final rule of derivation $\mathcal{D}$. 
## Large-Step Semantics

<table>
<thead>
<tr>
<th>Rule</th>
<th>Logic</th>
<th>Type</th>
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<tbody>
<tr>
<td><strong>Skip</strong></td>
<td>$\langle \sigma, \text{skip} \rangle \downarrow \sigma$</td>
<td><strong>ASSGN</strong></td>
<td>$\langle \sigma, a \rangle \downarrow n$  $\langle \sigma, x := a \rangle \downarrow \sigma[x \mapsto n]$</td>
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<tr>
<td><strong>SEQ</strong></td>
<td>$\langle \sigma, c_1 \rangle \downarrow \sigma'$  $\langle \sigma', c_2 \rangle \downarrow \sigma''$</td>
<td></td>
<td>$\langle \sigma, c_1; c_2 \rangle \downarrow \sigma''$</td>
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<tr>
<td><strong>IF-T</strong></td>
<td>$\langle \sigma, b \rangle \downarrow \text{true}$</td>
<td><strong>IF-F</strong></td>
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<td>$\langle \sigma, b \rangle \downarrow \text{false}$</td>
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