Lecture 27
Records and Subtyping
Records

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**Example:**

\[
\{ \text{foo} = 32, \text{bar} = \text{true} \}
\]

is a record value with an integer field `foo` and a boolean field `bar`. 
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*Records* are a generalization of tuples where we mark each field with a label.

**Example:**

```{foo = 32, bar = true}
```

is a record value with an integer field `foo` and a boolean field `bar`.

Its type is:

```{foo : int, bar : bool}```
Syntax

\[ l \in \mathcal{L} \]
\[ e ::= \cdots \mid \{l_1 = e_1, \ldots, l_n = e_n\} \mid e.l \]
\[ v ::= \cdots \mid \{l_1 = v_1, \ldots, l_n = v_n\} \]
\[ \tau ::= \cdots \mid \{l_1: \tau_1, \ldots, l_n: \tau_n\} \]
Dynamic Semantics

\[ E ::= \ldots \]
\[ \mid \{ l_1 = v_1, \ldots, l_{i-1} = v_{i-1}, l_i = E, l_{i+1} = e_{i+1}, \ldots, l_n = e_n \} \]
\[ \mid E.l \]

\[ \{ l_1 = v_1, \ldots, l_n = v_n \}.l_i \rightarrow v_i \]
∀\(i \in 1..n\). \(\Gamma \vdash e_i : \tau_i\)

\[\Gamma \vdash \{l_1 = e_1, \ldots, l_n = e_n\} : \{l_1 : \tau_1, \ldots, l_n : \tau_n\}\]

\[\Gamma \vdash e : \{l_1 : \tau_1, \ldots, l_n : \tau_n\}\]

\[\Gamma \vdash e.l_i : \tau_i\]
Example

\[
\text{GETX} \triangleq \lambda p : \{x : \text{int}, y : \text{int}\}. p.x
\]
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\[ \text{GETX} \{ x = 4, y = 2 \} \]
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\( \text{GETX} \{ x = 4, y = 2, z = 42 \} \)
Example

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\]

\[
\text{GETX} \{x = 4, y = 2\}
\]

\[
\text{GETX} \{x = 4, y = 2, z = 42\}
\]

\[
\text{GETX} \{y = 2, x = 4\}
\]
Subtyping

Definition (Subtype)

$\tau_1$ is a subtype of $\tau_2$, written $\tau_1 \leq \tau_2$, if a program can use a value of type $\tau_1$ whenever it would use a value of type $\tau_2$.

If $\tau_1 \leq \tau_2$, we also say $\tau_2$ is the supertype of $\tau_1$. 
Subtyping

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If $\tau_1 \leq \tau_2$, we also say $\tau_2$ is the supertype of $\tau_1$.

\[
\Gamma \vdash e : \tau \quad \tau \leq \tau' \\
\frac{}{\Gamma \vdash e : \tau'} \quad \text{SUBSUMPTION}
\]

This typing rule says that if $e$ has type $\tau$ and $\tau$ is a subtype of $\tau'$, then $e$ also has type $\tau'$. 
Record Subtyping

We’ll define a new subtyping relation that works together with the subsumption rule.

\[ \tau_1 \leq \tau_2 \]
Record Subtyping

This program isn’t well-typed (yet):

\[(\lambda p : \{x : \text{int}\}. p.x) \{x = 4, y = 2\}\]
Record Subtyping

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(\lambda p : \{x : \text{int}\}. \ p.x) \ \{ x = 4, y = 2 \}
\]

So let’s add width subtyping:

\[
k \geq 0 \\
\{ l_1 : \tau_1, \ldots, l_{n+k} : \tau_{n+k} \} \leq \{ l_1 : \tau_1, \ldots, l_n : \tau_n \}
\]
Record Subtyping

This program also doesn’t get stuck:

$$(\lambda p:\{x: \text{int}, y: \text{int}\}. p.x + p.y) \{y = 37, x = 5\}$$
Record Subtyping

This program also doesn’t get stuck:

\[(\lambda p : \{ x : \textbf{int}, y : \textbf{int} \}. p.x + p.y) \{ y = 37, x = 5 \}\]

So we can make it well-typed by adding permutation subtyping:

\[
\pi \text{ is a permutation on } 1..n \\
\{ l_1 : \tau_1, \ldots, l_n : \tau_n \} \leq \{ l_{\pi(1)} : \tau_{\pi(1)}, \ldots, l_{\pi(n)} : \tau_{\pi(n)} \} 
\]
Record Subtyping

Does this program get stuck? Is it well-typed?

\[ (\lambda p : \{ x : \{ y : \texttt{int} \} \}. p.x.y) \{ x = \{ y = 4, z = 2 \} \} \]
Record Subtyping

Does this program get stuck? Is it well-typed?

\[(\lambda p : \{ x : \{ y : \text{int} \} \}. p . x . y) \{ x = \{ y = 4, z = 2 \} \}\]

Let’s add depth subtyping:

\[\forall i \in 1..n. \quad \tau_i \leq \tau'_i\]

\[\{ l_1 : \tau_1, \ldots, l_n : \tau_n \} \leq \{ l_1 : \tau'_1, \ldots, l_n : \tau'_n \}\]
Record Subtyping

Putting all three forms of record subtyping together:

$$\forall i \in 1..n. \exists j \in 1..m. \ l'_i = l_j \land \tau_j \leq \tau'_i \quad \frac{\{l_1:\tau_1, \ldots, l_m:\tau_m\} \leq \{l'_1:\tau'_1, \ldots, l'_n:\tau'_n\}}{\text{S-RECORD}}$$
Standard Subtyping Rules

We always make the subtyping relation both reflexive and transitive.

\[ \tau \leq \tau \quad \text{S-REFL} \]

\[ \frac{\tau_1 \leq \tau_2 \quad \tau_2 \leq \tau_3}{\tau_1 \leq \tau_3} \quad \text{S-TRANS} \]

Think of every type describing a set of values. Then \( \tau_1 \leq \tau_2 \) when \( \tau_1 \)'s values are a subset of \( \tau_2 \)'s.
Top Type

It’s sometimes useful to define a *maximal* type with respect to subtyping:

\[ \tau ::= \cdots | \top \]

\[ \frac{}{\tau \leq \top} \text{ S-Top} \]

Everything is a subtype of \( \top \), as in Java’s `Object` or Go’s `interface{}`.
We can also write subtyping rules for sums and products:

\[
\frac{\tau_1 \leq \tau'_1 \quad \tau_2 \leq \tau'_2}{\tau_1 + \tau_2 \leq \tau'_1 + \tau'_2} \quad \text{S-Sum}
\]
We can also write subtyping rules for sums and products:

**S-SUM**

\[
\frac{\tau_1 \leq \tau'_1 \quad \tau_2 \leq \tau'_2}{\tau_1 + \tau_2 \leq \tau'_1 + \tau'_2}
\]

**S-PRODUCT**

\[
\frac{\tau_1 \leq \tau'_1 \quad \tau_2 \leq \tau'_2}{\tau_1 \times \tau_2 \leq \tau'_1 \times \tau'_2}
\]
Function Types

How should we decide whether one function type is a subtype of another?

\[
\begin{align*}
\frac{\tau_1 \to \tau_2 \leq \tau_1' \to \tau_2'}{\text{S-FUNCTION}}
\end{align*}
\]
Desiderata

We’d like to have:

\[
\text{int} \rightarrow \{x : \text{int}, y : \text{int}\} \leq \text{int} \rightarrow \{x : \text{int}\}
\]
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\[
\text{int} \rightarrow \{x:\text{int}, y:\text{int}\} \leq \text{int} \rightarrow \{x:\text{int}\}
\]

And:

\[
\{x:\text{int}\} \rightarrow \text{int} \leq \{x:\text{int}, y:\text{int}\} \rightarrow \text{int}
\]
Desiderata

We’d like to have:

$$\text{int} \rightarrow \{x: \text{int}, y: \text{int}\} \leq \text{int} \rightarrow \{x: \text{int}\}$$

And:

$$\{x: \text{int}\} \rightarrow \text{int} \leq \{x: \text{int}, y: \text{int}\} \rightarrow \text{int}$$

In general, to prove:

$$\tau_1 \rightarrow \tau_2 \leq \tau_1' \rightarrow \tau_2'$$

we’ll require:

- Argument types are contravariant: $$\tau_1' \leq \tau_1$$
- Return types are covariant: $$\tau_2 \leq \tau_2'$$
Function Subtyping

Putting these two pieces together, we get the subtyping rule for function types:

\[
\frac{\tau'_1 \leq \tau_1 \quad \tau_2 \leq \tau'_2}{\tau_1 \rightarrow \tau_2 \leq \tau'_1 \rightarrow \tau'_2} \quad \text{S-FUNCTION}
\]
Reference Subtyping

What should the relationship be between $\tau$ and $\tau'$ in order to have $\tau \text{ ref} \leq \tau' \text{ ref}$?
Example

If $r'$ has type $\tau' \textbf{ref}$, then $!r'$ has type $\tau'$.

Imagine we replace $r'$ with $r$, where $r$ has a type $\tau \textbf{ref}$ that we’ve somehow decided is a subtype of $\tau' \textbf{ref}$. 
Example

If $r'$ has type $\tau' \text{ ref}$, then $!r'$ has type $\tau'$.

Imagine we replace $r'$ with $r$, where $r$ has a type $\tau \text{ ref}$ that we’ve somehow decided is a subtype of $\tau' \text{ ref}$.

Then $!r$ should still produce something can be treated as a $\tau'$. In other words, it should have a type that is a subtype of $\tau'$.

So the referent type should be covariant:

\[
\tau \leq \tau' \\
\frac{}{\tau \text{ ref} \leq \tau' \text{ ref}}
\]
Example

If $v$ has type $\tau'$, then $r' := v$ should be legal.

If we replace $r'$ with $r$, then it must still be legal to assign $r := v$. So $!r$ would then produce a value of type $\tau'$. 
Example

If \( v \) has type \( \tau' \), then \( r' := v \) should be legal.

If we replace \( r' \) with \( r \), then it must still be legal to assign \( r := v \). So \( !r \) would then produce a value of type \( \tau' \).

So the referent type should be contravariant!

\[
\frac{\tau' \leq \tau}{\tau \text{ ref} \leq \tau' \text{ ref}}
\]
In fact, subtyping for reference types must be \emph{invariant}: a reference type $\tau \text{ ref}$ is a subtype of $\tau' \text{ ref}$ if and only if $\tau \leq \tau'$ and $\tau' \leq \tau$.

\[
\frac{\tau \leq \tau'}{\tau \text{ ref} \leq \tau' \text{ ref}} \quad \text{S-REF}
\]
Tragically, Java’s mutable arrays use covariant subtyping!
Java Arrays

Tragically, Java’s mutable arrays use covariant subtyping!

Suppose that Cow is a subtype of Animal.

Code that only reads from arrays typechecks:

```java
Animal[ ] arr = new Cow[ ] { new Cow(“Alfonso”) };
Animal a = arr[0];
```
Tragically, Java’s mutable arrays use covariant subtyping!

Suppose that Cow is a subtype of Animal.

Code that only reads from arrays typechecks:

```java
Animal[ ] arr = new Cow[ ] { new Cow("Alfonso") };  
Animal a = arr[0];
```

but writing to the array can get into trouble:

```java
arr[0] = new Animal("Brunhilda");
```

Specifically, this generates an ArrayStoreException.