Lecture 19
Continuations
Continuations

In the preceding translations, the control structure of the source language was translated directly into the corresponding control structure in the target language.

For example:

\[ \mathcal{T}[\lambda x. e] = \lambda x. \mathcal{T}[e] \]

\[ \mathcal{T}[e_1 e_2] = \mathcal{T}[e_1] \mathcal{T}[e_2] \]

What can go wrong with this approach?
Continuations

- A snippet of code that represents “the rest of the program”
- Can be used directly by programmers...
- ...or in program transformations by a compiler
- Make the control flow of the program explicit
- Also useful for defining the meaning of features like exceptions
Example

Consider the following expression:

$$(\lambda x. x) \left( (1 + 2) + 3 \right) + 4$$
Example

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$$(\lambda x. x) ((1 + 2) + 3) + 4$$

If we make all of the continuations explicit, we obtain:

$$k_0 = \lambda v. (\lambda x. x) v$$
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$$k_0 = \lambda v. (\lambda x. x) v$$
$$k_1 = \lambda a. k_0 (a + 4)$$
Example

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If we make all of the continuations explicit, we obtain:

$$k_0 = \lambda v. (\lambda x. x) v$$

$$k_1 = \lambda a. k_0 (a + 4)$$

$$k_2 = \lambda b. k_1 (b + 3)$$
Consider the following expression:

\[(\lambda x. x) \ ((1 + 2) + 3) + 4\]

If we make all of the continuations explicit, we obtain:

\[k_0 = \lambda v. (\lambda x. x) \ v\]
\[k_1 = \lambda a. k_0 \ (a + 4)\]
\[k_2 = \lambda b. k_1 \ (b + 3)\]
\[k_3 = \lambda c. k_2 \ (c + 2)\]
Example

Consider the following expression:

$$(\lambda x. x) \ ((1 + 2) + 3) + 4$$

If we make all of the continuations explicit, we obtain:

$$k_0 = \lambda v. \ (\lambda x. x) \ v$$
$$k_1 = \lambda a. \ k_0 \ (a + 4)$$
$$k_2 = \lambda b. \ k_1 \ (b + 3)$$
$$k_3 = \lambda c. \ k_2 \ (c + 2)$$

The original expression is equivalent to $k_3 \ 1$, or:

$$(\lambda c. \ (\lambda b. \ (\lambda a. \ (\lambda v. \ (\lambda x. x) \ v) \ (a + 4)) \ (b + 3)) \ (c + 2)) \ 1$$
Recall that let $x = e$ in $e'$ is syntactic sugar for $(\lambda x. e') e$.

Hence, we can rewrite the expression with continuations more succinctly as

$$
\begin{align*}
\text{let } c &= 1 \text{ in } \\
\text{let } b &= c + 2 \text{ in } \\
\text{let } a &= b + 3 \text{ in } \\
\text{let } v &= a + 4 \text{ in } \\
(\lambda x. x) v
\end{align*}
$$
We write $CPS[e] k = \ldots$ instead of $CPS[e] = \lambda k. \ldots$

We assume that the new variables introduced are “fresh.”
CPS Transformation

\[ \text{CPS}[n] k = kn \]

We write \( \text{CPS}[e] k = \ldots \) instead of \( \text{CPS}[e] = \lambda k. \ldots \)

We assume that the new variables introduced are “fresh.”
CPS Transformation

\[
\begin{align*}
\text{CPS}[n] k &= k n \\
\text{CPS}[e_1 + e_2] k &= \text{CPS}[e_1] (\lambda n. \text{CPS}[e_2] (\lambda m. k (n + m)))
\end{align*}
\]

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CPS Transformation

\[
\begin{align*}
CPS[n] \ k &= k \ n \\
CPS[e_1 + e_2] \ k &= CPS[e_1] \ (\lambda n. CPS[e_2] \ (\lambda m. k \ (n + m))) \\
CPS[(e_1, e_2)] \ k &= CPS[e_1] \ (\lambda v. CPS[e_2] \ (\lambda w. k \ (v, w)))
\end{align*}
\]

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\text{CPS}[n] k &= k n \\
\text{CPS}[e_1 + e_2] k &= \text{CPS}[e_1] (\lambda n. \text{CPS}[e_2] (\lambda m. k (n + m))) \\
\text{CPS}[(e_1, e_2)] k &= \text{CPS}[e_1] (\lambda v. \text{CPS}[e_2] (\lambda w. k (v, w))) \\
\text{CPS}[\#1 e] k &= \text{CPS}[e] (\lambda v. k (\#1 v)) 
\end{align*}
\]

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CPS Transformation

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\begin{align*}
\text{CPS}[n] k &= kn \\
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\text{CPS}[(e_1, e_2)] k &= \text{CPS}[e_1] (\lambda v. \text{CPS}[e_2] (\lambda w. k (v, w))) \\
\text{CPS}[^1 e] k &= \text{CPS}[e] (\lambda v. k (^1 v)) \\
\text{CPS}[^2 e] k &= \text{CPS}[e] (\lambda v. k (^2 v))
\end{align*}
\]

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CPS Transformation

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CPS[n] k &= k n \\
CPS[e_1 + e_2] k &= CPS[e_1] (\lambda n. CPS[e_2] (\lambda m. k (n + m))) \\
CPS[(e_1, e_2)] k &= CPS[e_1] (\lambda v. CPS[e_2] (\lambda w. k (v, w))) \\
CPS[#1 e] k &= CPS[e] (\lambda v. k (#1 v)) \\
CPS[#2 e] k &= CPS[e] (\lambda v. k (#2 v)) \\
CPS[x] k &= k x
\end{align*}
\]

We write \(CPS[e] k = \ldots\) instead of \(CPS[e] = \lambda k. \ldots\)

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CPS Transformation

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\begin{align*}
\text{CPS}[n] k &= kn \\
\text{CPS}[e_1 + e_2] k &= \text{CPS}[e_1] (\lambda n. \text{CPS}[e_2] (\lambda m. k (n + m))) \\
\text{CPS}[(e_1, e_2)] k &= \text{CPS}[e_1] (\lambda v. \text{CPS}[e_2] (\lambda w. k (v, w))) \\
\text{CPS}[\#1 e] k &= \text{CPS}[e] (\lambda v. k (#1 v)) \\
\text{CPS}[\#2 e] k &= \text{CPS}[e] (\lambda v. k (#2 v)) \\
\text{CPS}[x] k &= kx \\
\text{CPS}[\lambda x. e] k &= k (\lambda x. \lambda k'. \text{CPS}[e] k')
\end{align*}
\]

We write \( \text{CPS}[e] k = \ldots \) instead of \( \text{CPS}[e] = \lambda k. \ldots \)

We assume that the new variables introduced are “fresh.”
**CPS Transformation**

\[
\text{CPS}[n]\ k = kn
\]

\[
\text{CPS}[e_1 + e_2]\ k = \text{CPS}[e_1] (\lambda n. \text{CPS}[e_2] (\lambda m. k (n + m)))
\]

\[
\text{CPS}[(e_1, e_2)]\ k = \text{CPS}[e_1] (\lambda v. \text{CPS}[e_2] (\lambda w. k (v, w)))
\]

\[
\text{CPS}[#1 e]\ k = \text{CPS}[e] (\lambda v. k (#1 v))
\]

\[
\text{CPS}[#2 e]\ k = \text{CPS}[e] (\lambda v. k (#2 v))
\]

\[
\text{CPS}[x]\ k = kx
\]

\[
\text{CPS}[\lambda x. e]\ k = k (\lambda x. \lambda k'. \text{CPS}[e] k')
\]

\[
\text{CPS}[e_1 e_2]\ k = \text{CPS}[e_1] (\lambda f. \text{CPS}[e_2] (\lambda v. f v k))
\]

We write \(\text{CPS}[e]\ k = \ldots\) instead of \(\text{CPS}[e] = \lambda k. \ldots\)

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