Lecture 18
Evaluation Contexts and
Definitional Translation
Review: Call-by-Value

Here are the syntax and CBV semantics of $\lambda$-calculus:

$$e ::= x \mid \lambda x.\ e \mid e_1\ e_2$$
$$v ::= \lambda x.\ e$$

$$e_1 \rightarrow e'_1 \quad e \rightarrow e'$$
$$\frac{e_1\ e_2 \rightarrow e'_1\ e_2}{v\ e \rightarrow v\ e'}$$

$(\lambda x.\ e)\ v \rightarrow e\{v/x\}^\beta$

There are two kinds of rules: congruence rules that specify evaluation order and computation rules that specify the “interesting” reductions.
Evaluation contexts let us separate out these two kinds of rules.
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An evaluation context $E$ is an expression with a “hole” in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= [\cdot] \mid E \ e \mid \nu \ E$$
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$$E ::= [\cdot] \mid E \, e \mid \nu E$$

We write $E[e]$ to mean the evaluation context $E$ where the hole has been replaced with the expression $e$. 
Examples

\[ E_1 = [\cdot] (\lambda x. x) \]
\[ E_1[\lambda y. y y] = (\lambda y. y y) \lambda x. x \]
Examples

\[ E_1 = [\cdot] (\lambda x. x) \]
\[ E_1[\lambda y. yy] = (\lambda y. yy) \lambda x. x \]

\[ E_2 = (\lambda z. zz) [\cdot] \]
\[ E_2[\lambda x. \lambda y. x] = (\lambda z. zz) (\lambda x. \lambda y. x) \]
Examples

\[ E_1 = [\cdot] (\lambda x. x) \]

\[ E_1[\lambda y. yy] = (\lambda y. yy) \lambda x. x \]

\[ E_2 = (\lambda z. zz) [\cdot] \]

\[ E_2[\lambda x. \lambda y. x] = (\lambda z. zz) (\lambda x. \lambda y. x) \]

\[ E_3 = ([\cdot] \lambda x. xx) ((\lambda y. y) (\lambda y. y)) \]

\[ E_3[\lambda f. \lambda g. fg] = ((\lambda f. \lambda g. fg) \lambda x. xx) ((\lambda y. y) (\lambda y. y)) \]
With evaluation contexts, we can define the evaluation semantics for the CBV \( \lambda \)-calculus with just two rules: one for evaluation contexts, and one for \( \beta \)-reduction.

\[
E ::= \begin{cases} \emptyset & \text{if } E \vdash e \Downarrow \text{true} \\ \left[ e \right] E & \text{if } E \vdash e \Downarrow \text{false} \\ vE & \text{if } E \vdash (\lambda x. e) v \Downarrow e[v/x] \\
\end{cases}
\]
CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV $\lambda$-calculus with just two rules: one for evaluation contexts, and one for $\beta$-reduction.

With this syntax:

$$E ::= [\cdot] \mid E \, e \mid \nu \, E$$

The small-step rules are:

$$e \to e' \quad \frac{E[e] \to E[e']} {E[e] \to E[e']}$$

$$\beta \quad (\lambda x. e) \, v \to e\{v/x\}$$
We can also define the semantics of CBN $\lambda$-calculus with evaluation contexts.
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For call-by-name, the syntax for evaluation contexts is different:

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For call-by-name, the syntax for evaluation contexts is different:

$$E ::= [\cdot] \mid E\ e$$

But the small-step rules are the same:

$$e \rightarrow e'$$

$$\frac{}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x. e)\ e' \rightarrow e\{e'/x\}}$$
We know how to encode Booleans, conditionals, natural numbers, and recursion in $\lambda$-calculus.

Can we define a real programming language by translating everything in it into the $\lambda$-calculus?
Definitional Translation

We know how to encode Booleans, conditionals, natural numbers, and recursion in $\lambda$-calculus.

Can we define a *real* programming language by translating everything in it into the $\lambda$-calculus?

In *definitional translation*, we define a denotational semantics where the target is a simpler programming language instead of mathematical objects.
Multi-Argument $\lambda$-calculus

Let’s define a version of the $\lambda$-calculus that allows functions to take multiple arguments.

\[ e ::= x \mid \lambda x_1, \ldots, x_n. e \mid e_0 \, e_1 \, \ldots \, e_n \]
Multi-Argument $\lambda$-calculus

We can define a CBV operational semantics:

$$E ::= [\cdot] \mid v_0 \ldots v_{i-1} E e_{i+1} \ldots e_n$$

$$e \rightarrow e' \quad \frac{E[e] \rightarrow E[e']} {E[e] \rightarrow E[e']}$$

$$\left( \lambda x_1, \ldots, x_n. e_0 \right) v_1 \ldots v_n \rightarrow e_0 \{ v_1/x_1 \} \{ v_2/x_2 \} \ldots \{ v_n/x_n \}$$

The evaluation contexts ensure that we evaluate multi-argument applications $e_0 e_1 \ldots e_n$ from left to right.
Definitional Translation

The multi-argument $\lambda$-calculus isn’t any more expressive than the pure $\lambda$-calculus.
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We can define a translation $T[\cdot]$ that takes an expression in the multi-argument $\lambda$-calculus and returns an equivalent expression in the pure $\lambda$-calculus.
Definitional Translation

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We can define a translation $T[\cdot]$ that takes an expression in the multi-argument $\lambda$-calculus and returns an equivalent expression in the pure $\lambda$-calculus.

$$
T[x] = x \\
T[\lambda x_1, \ldots, x_n. e] = \lambda x_1. \ldots \lambda x_n. T[e] \\
T[e_0 e_1 e_2 \ldots e_n] = (\ldots((T[e_0] T[e_1]) T[e_2]) \ldots T[e_n])
$$

This translation *curries* the multi-argument $\lambda$-calculus.
Products (Pairs) and Let

Syntax

\[ e ::= x \]
\[ | \lambda x. \, e \]
\[ | e_1 \, e_2 \]
\[ | (e_1, e_2) \]
\[ | \#1 \, e \]
\[ | \#2 \, e \]
\[ | \text{let } x = e_1 \text{ in } e_2 \]

\[ v ::= \lambda x. \, e \]
\[ | (v_1, v_2) \]
Evaluation Contexts

\[ E ::= [\cdot] \]
\[ | E e \]
\[ | \nu E \]
\[ | (E, e) \]
\[ | (\nu, E) \]
\[ | \#1 E \]
\[ | \#2 E \]
\[ | \text{let } x = E \text{ in } e_2 \]
Products (Pairs) and Let

Semantics

\[ e \rightarrow e' \]
\[ \frac{E[e] \rightarrow E[e']}{} \]

\[ (\lambda x. e) v \rightarrow e\{v/x\} \]

\[ \#1 (v_1, v_2) \rightarrow v_1 \]
\[ \#2 (v_1, v_2) \rightarrow v_2 \]

\[ \text{let } x = v \text{ in } e \rightarrow e\{v/x\} \]
Products (Pairs) and Let

Translation

\[ \mathcal{T}[x] = x \]
\[ \mathcal{T}[^\lambda x. e] = \lambda x. \mathcal{T}[e] \]
\[ \mathcal{T}[e_1 e_2] = \mathcal{T}[e_1] \mathcal{T}[e_2] \]
\[ \mathcal{T}[\langle e_1, e_2 \rangle] = (\lambda x. \lambda y. \lambda f. f x y) \mathcal{T}[e_1] \mathcal{T}[e_2] \]
\[ \mathcal{T}[\#1 e] = \mathcal{T}[e] (\lambda x. \lambda y. x) \]
\[ \mathcal{T}[\#2 e] = \mathcal{T}[e] (\lambda x. \lambda y. y) \]
\[ \mathcal{T}[\text{let } x = e_1 \text{ in } e_2] = (\lambda x. \mathcal{T}[e_2]) \mathcal{T}[e_1] \]
Laziness

Consider the call-by-name $\lambda$-calculus...

Syntax

$$e ::= x$$

$$| e_1 e_2$$

$$| \lambda x. e$$

$$v ::= \lambda x. e$$

Semantics

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2}$$

$$\frac{e_{1} \rightarrow e'_{1}}{(\lambda x. e_1) e_2 \rightarrow e_1\{e_2/x\}}^\beta$$
Laziness

Translation

\[ T[x] = x (\lambda y. y) \]
\[ T[\lambda x. e] = \lambda x. T[e] \]
\[ T[e_1 e_2] = T[e_1] (\lambda z. T[e_2]) \quad \text{if } z \text{ is not a free variable of } e_2 \]
Syntax

\[ e ::= x \]
\[ \mid \lambda x. e \]
\[ \mid e_0 e_1 \]

\[ \nu ::= \lambda x. e \]
Syntax

\[ e ::= x \]
\[ \quad | \quad \lambda x.\ e \]
\[ \quad | \quad e_0\ e_1 \]
\[ \quad | \quad \text{ref } e \]

\[ \nu ::= \lambda x.\ e \]
Syntax

\[
e ::= x \\
\quad \mid \lambda x. e \\
\quad \mid e_0 e_1 \\
\quad \mid \text{ref } e \\
\quad \mid !e
\]

\[
v ::= \lambda x. e
\]
Syntax

\[ e ::= x \]
\[ \quad | \lambda x. e \]
\[ \quad | e_0 \ e_1 \]
\[ \quad | \text{ref } e \]
\[ \quad | !e \]
\[ \quad | e_1 ::= e_2 \]

\[ \nu ::= \lambda x. e \]
Syntax

\[ e ::= x \]
\[ \quad | \lambda x. e \]
\[ \quad | e_0 e_1 \]
\[ \quad | \text{ref } e \]
\[ \quad | !e \]
\[ \quad | e_1 ::= e_2 \]
\[ \quad | \ell \]

\[ \nu ::= \lambda x. e \]
Syntax

\[ e ::= x \]
\[ \mid \lambda x. e \]
\[ \mid e \_ e_1 \]
\[ \mid \text{ref } e \]
\[ \mid !e \]
\[ \mid e_1 ::= e_2 \]
\[ \mid \ell \]

\[ \nu ::= \lambda x. e \]
\[ \mid \ell \]
References

Evaluation Contexts

\[ E ::= [\cdot] \]
\[ \mid E \, e \]
\[ \mid v \, E \]
References

Evaluation Contexts

\[
E ::= [\cdot] \\
| E e \\
| \nu E \\
| \text{ref } E
\]
Evaluation Contexts

\[ E ::= \cdot \]
\[ | E \ e \]
\[ | v \ E \]
\[ | \text{ref} \ E \]
\[ | !E \]
Evaluation Contexts

\[ E ::= [\cdot] \]
\[ \mid E \ e \]
\[ \mid \nu \ E \]
\[ \mid \text{ref} \ E \]
\[ \mid !E \]
\[ \mid E ::= e \]
Evaluation Contexts

\[ E ::= [\cdot] \]

\[ | E e \]

\[ | v E \]

\[ | \text{ref } E \]

\[ | !E \]

\[ | E ::= e \]

\[ | v ::= E \]
References

Semantics

\[
\begin{align*}
\langle \sigma, e \rangle & \rightarrow \langle \sigma', e' \rangle \\
\langle \sigma, E[e] \rangle & \rightarrow \langle \sigma', E[e'] \rangle \\
\ell \notin \text{dom}(\sigma) & \quad \rightarrow \langle \sigma[\ell \mapsto v], \ell \rangle \\
\langle \sigma, \text{ref} \ v \rangle & \rightarrow \langle \sigma[\ell \mapsto v], \ell \rangle \\
\langle \sigma, \lambda x. e \ v \rangle & \rightarrow \langle \sigma, e\{v/x\} \rangle^\beta \\
\langle \sigma, \ell \rangle & \rightarrow \langle \sigma, v \rangle \\
\langle \sigma, \ell := v \rangle & \rightarrow \langle \sigma[\ell \mapsto v], v \rangle
\end{align*}
\]
Translation

...left as an exercise to the reader. ;-)
Adequacy

How do we know if a translation is correct?
Adequacy

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Every target evaluation should represent a source evaluation...

**Definition (Soundness)**

\[ \forall e \in \text{Exp}_{\text{src}}. \text{ if } T[e] \rightarrow^{*}_{\text{trg}} v' \text{ then } \exists v. e \rightarrow^{*}_{\text{src}} v \]

and \( v' \) equivalent to \( v \)
Adequacy

How do we know if a translation is correct?

Every target evaluation should represent a source evaluation...

**Definition (Soundness)**

\[ \forall e \in \text{Exp}_{\text{src}} \cdot \text{if } \mathcal{T}[e] \xrightarrow{\ast}_{\text{trg}} \nu' \text{ then } \exists \nu. \ e \xrightarrow{\ast}_{\text{src}} \nu \]

and \( \nu' \) equivalent to \( \nu \)

...and every source evaluation should have a target evaluation:

**Definition (Completeness)**

\[ \forall e \in \text{Exp}_{\text{src}} \cdot \text{if } e \xrightarrow{\ast}_{\text{src}} \nu \text{ then } \exists \nu'. \mathcal{T}[e] \xrightarrow{\ast}_{\text{trg}} \nu' \]

and \( \nu' \) equivalent to \( \nu \)