1

```
R := 0;
while ( 0 < X ) do
    R := R + X ;
    X := X - 1
```

c

1

```
R := 0;
while (0 < X) do
    R := R + X;
    X := X - 1
```

Prop. $\forall \sigma, \sigma', n.\ \ \sigma(X) = n \ \wedge$

$$\langle \sigma, c \rangle \Downarrow \sigma' \Longrightarrow$$

$$\sigma'(R) = \sum_{i=1}^{n} i$$

Proof.

c

① $R := 0;$
c_{loop} while ($0 < X$) do
$R := R + X;$
$X := X - 1$

Prop. $P(n) \triangleq \forall \sigma, \sigma'.\ \sigma(X) = n \ \wedge$
$\langle \sigma, c \rangle \Downarrow \sigma' \Longrightarrow$
$\sigma'(R) = \sum_{i=1}^{n} i$

$\forall n.\ P(n).$

Proof.

By induction on n.

Base case

$n = 0$

Assume $\sigma(X) = n$
$\langle \sigma, c \rangle \Downarrow \sigma'$

To show $\sigma'(R) = \sum_{i=1}^{n} i = 0$

From $\langle \sigma, c \rangle \Downarrow \sigma'$, we know Seq. was used.

① $\langle \sigma, R := 0 \rangle \Downarrow \sigma_1$

② $\langle \sigma_1, c_{loop} \rangle \Downarrow \sigma'$

From ①, we know Assign.
So $\sigma_1 = \sigma[R \mapsto 0]$

From

c

① $R := 0;$

c_{loop}:

```
while ( 0 < X ) do
    R := R + X ;
    X := X - 1
```

Prop. $P(n) \triangleq \forall \sigma, \sigma'.\ \sigma(X) = n \ \wedge$

$$\langle \sigma, c \rangle \Downarrow \sigma' \implies \sigma'(R) = \sum_{i=1}^{n} i$$

$\forall n.\ P(n).$

Proof.

By induction on n.

Base case

$n = 0$

Assume $\sigma(X) = 0$

$\langle \sigma, c \rangle \Downarrow \sigma'$

To show $\sigma'(R) = \sum_{i=1}^{n} i = 0$

From $\langle \sigma, c \rangle \Downarrow \sigma'$, we know Seq. was used.

① $\langle \sigma, R := 0 \rangle \Downarrow \sigma_1$

② $\langle \sigma_1, c_{loop} \rangle \Downarrow \sigma'$

From ①, we know Assign.

So $\sigma_1 = \sigma[R \mapsto 0]$

Observe $\langle \sigma_1, 0 < X \rangle \Downarrow \text{false}$.

From ② we know While-F was used.

$\sigma' = \sigma'$

$= \sigma[R \mapsto 0]$.

$\sigma'(R) = 0$ ✓✓

c

```
① R := 0;
c_loop  while ( 0 < X ) do
          R := R + X;
          X := X - 1
```

<u>Prop.</u> $P(n) \triangleq \forall \sigma, \sigma'.\ \sigma(X) = n \land \langle \sigma, c \rangle \Downarrow \sigma' \Rightarrow \sigma'(R) = \sum_{i=1}^{n} i$.

$\forall n.\ P(n)$.

<u>Proof.</u>

By induction on n.

<u>Inductive case</u>

$n = m + 1$

Assume $\sigma(X) = n$

① $\langle \sigma, c \rangle \Downarrow \sigma'$

From ①, Seq was used.

② $\langle \sigma, R := 0 \rangle \Downarrow \sigma_1 = \sigma[R \mapsto 0]$

③ $\langle \sigma_1, c_{loop} \rangle \Downarrow \sigma'$

Observe $\langle \sigma_1, 0 < X \rangle \Downarrow true$.

From ③, While-T was used.

$\langle \sigma_1, c \rangle \Downarrow \sigma_2$

$\langle \sigma_2, c_{loop} \rangle \Downarrow \sigma'$

c

① $R := 0;$

c_{loop} $\quad$ while $(0 < X)$ do

$\quad R := R + X;$

$\quad X := X - 1$

Prop. $P(n) \triangleq \forall \sigma, \sigma'.\ \sigma(X) = n \wedge \langle \sigma, c \rangle \Downarrow \sigma' \Rightarrow \sigma'(R) = \sum_{i=1}^{n} i.$

$\forall n.\ P(n).$

Proof.

By induction on n.

Inductive case

$n = m + 1$

Assume $\sigma(X) = n$

① $\langle \sigma, c \rangle \Downarrow \sigma'$.

From ①, Seq was used.

② $\langle \sigma, R := 0 \rangle \Downarrow \sigma_1 = \sigma[R \mapsto 0]$

③ $\langle \sigma_1, c_{loop} \rangle \Downarrow \sigma'$

Observe $\langle \sigma_1, 0 < X \rangle \Downarrow \mathrm{true}$.

From ③, While-T was used.

④ $\langle \sigma_1, R := R + X;\ X := X - 1 \rangle \Downarrow \sigma_2$

⑤ $\langle \sigma_2, c_{loop} \rangle \Downarrow \sigma'$

Observe $\sigma_2(X) = m$.

c

① $R := 0;$

c_{loop}

while (0 < X) do
  R := R + X ;
  X := X − 1

Prop. $P(n) \triangleq \forall \sigma, \sigma'.\ \sigma(X) = n \wedge \sigma(R) = m$

$\langle \sigma, c_{loop} \rangle \Downarrow \sigma' \implies$

$\forall n.\ P(n).$

$$\sigma'(R) = \sum_{i=1}^{n} i$$

Proof.

By induction on n.

Inductive case

$n = m + 1$

Assume $\sigma(X) = n$

① $\langle \sigma, c \rangle \Downarrow \sigma'$.

From ①, Seq was used.

② $\langle \sigma, R := 0 \rangle \Downarrow \sigma_1 = \sigma[R \mapsto 0]$

③ $\langle \sigma_1, c_{loop} \rangle \Downarrow \sigma'$

Observe $\langle \sigma_1, 0 < X \rangle \Downarrow$ true.

From ③, While-T was used.

④ $\langle \sigma_1, R := R + X;\ X := X - 1 \rangle \Downarrow \sigma_2$

⑤ $\langle \sigma_2, c_{loop} \rangle \Downarrow \sigma'$

Observe $\sigma_2(X) = m$.

c

① $R := 0;$

c_{loop}: while $(0 < X)$ do
  $R := R + X;$
  $X := X - 1$

Prop. $P(n) \triangleq \forall \sigma, \sigma'.\ \sigma(X) = n \land \sigma(R) = m$

$\forall n.\ P(n).$

$$\langle \sigma, c_{loop} \rangle \Downarrow \sigma' \implies \sigma'(R) = m + \sum_{i=1}^{n} i.$$

Proof.

By induction on n.

Inductive case

$n = m+1$

Assume $\sigma(X) = n$

① $\langle \sigma, c \rangle \Downarrow \sigma'$.

From ①, Seq was used.

② $\langle \sigma, R:=0 \rangle \Downarrow \sigma_1 = \sigma[R \mapsto 0]$

③ $\langle \sigma_1, c_{loop} \rangle \Downarrow \sigma'$

Observe $\langle \sigma_1, 0 < X \rangle \Downarrow \text{true}$.

From ③, While-T was used.

④ $\langle \sigma_1, R := R + X;\ X := X - 1 \rangle \Downarrow \sigma_2$

⑤ $\langle \sigma_2, c_{loop} \rangle \Downarrow \sigma'$

Observe $\sigma_2(X) = m$.

c

$$\text{①}\; R := 0;$$

$$c_{loop}: \quad \begin{array}{l} \texttt{while } (0 < X) \texttt{ do} \\ \quad R := R + X; \\ \quad X := X - 1 \end{array}$$

Prop. $P(n) \triangleq \forall \sigma, \sigma'.\ \sigma(X) = n \wedge \sigma(R) = m$

$$\langle \sigma, c_{loop} \rangle \Downarrow \sigma' \implies$$

$$\sigma'(R) = m + \sum_{i=1}^{n} i.$$

$\forall n.\ P(n).$

Proof.

By induction on n.

Base case

$n = 0$

Assume $\sigma(X) = 0$

$\sigma(R) = m$

① $\langle \sigma, c_{loop} \rangle \Downarrow \sigma'$

Observe $\langle \sigma, 0 < X \rangle \Downarrow$ false

From ① while-F.

$\sigma = \sigma'.$

$$\sigma'(R) = m = m + \sum_{i=1}^{0} i \quad \checkmark\checkmark$$

c

① $R := 0;$
c_{loop}: while $(0 < X)$ do
  $R := R + X;$
  $X := X - 1$

Prop. $P(n) \triangleq \forall \sigma, \sigma'.\ \sigma(X) = n \wedge \sigma(R) = m$

$\langle \sigma, c_{loop} \rangle \Downarrow \sigma' \implies$

$\sigma'(R) = m + \sum_{i=1}^{n} i.$

$\forall n.\ P(n).$

A1 A2 A3

Proof.

By induction on n.

Inductive case

$n = n' + 1$

Assume $\sigma(x) = n$
$\sigma(R) = m$

① $\langle \sigma, c_{loop} \rangle \Downarrow \sigma'$

Observe $\langle \sigma, 0 < X \rangle \Downarrow$ true.

From ① While-T was used.

② $\langle \sigma, R := R + X, X := X - 1 \rangle \Downarrow \sigma_1$

③ $\langle \sigma_1, c_{loop} \rangle \Downarrow \sigma'$

From ②, Seq, Assign rules.

A1' A2' $\sigma_1 = \sigma[R \mapsto m+n, X \mapsto n']$

Applying IH n'.

$\sigma'(R) = m + n + \sum_{i=1}^{n'} i = m + \sum_{i=1}^{n} i$ ✓

c

① $R := 0;$

c_{loop} $\boxed{\begin{array}{l}\texttt{while } (0 < X) \texttt{ do} \\ \quad R := R + X; \\ \quad X := X - 1\end{array}}$

Prop. $P(D).\ \forall \sigma, \sigma', n.\ \sigma(X) = n \wedge \sigma(R) = m$
$D \Vdash \langle \sigma, c_{loop} \rangle \Downarrow \sigma' \implies$
$\sigma'(R) = m + \sum_{i=1}^{n} i.$

$\forall D.\ P(D).$

Proof.

By induction on derivation of D.

Inductive case

$n = n' + 1$

Assume $\sigma(X) = n$
$\sigma(R) = m$

① $\langle \sigma, c_{loop} \rangle \Downarrow \sigma'$

Observe $\langle \sigma, 0 < X \rangle \Downarrow true.$

From ① While-T was used.

② $\langle \sigma, R := R + X, X := X - 1 \rangle \Downarrow \sigma_1$

③ $\langle \sigma_1, c_{loop} \rangle \Downarrow \sigma'$

From ②, Seq, Assign rules.

A1 A2 $\sigma_1 = \sigma[R \mapsto m + n, X \mapsto n']$

Applying IH n'.

$\sigma'(R) = m + n + \sum_{i=1}^{n'} i = m + \sum_{i=1}^{n} i$ ✓

c

①

```
R := 0;
while (0 < X) do
    R := R + X;
    X := X - 1
```

c_{loop}

Prop. $P(D).\ \forall \sigma, \sigma', n.\ \sigma(X) = n \wedge \sigma(R) = m$

$D \Vdash \langle \sigma, c_{loop} \rangle \Downarrow \sigma' \Longrightarrow$

$\sigma'(R) = m + \sum_{i=1}^{n} i.$

$\forall D.\ P(D).$

Proof.

By induction on derivation of D.

Case Skip

$c =$ skip.

Assume $\sigma(x) = n$

$\sigma(R) = m$

$D \Vdash \langle \sigma, c_{loop} \rangle \Downarrow \sigma'$

$c =$ while $(\ldots)$ do $\ldots$

$\cancel{\ }$

c

① $R := 0;$

c_{loop}: while $(0 < X)$ do
  $R := R + X;$
  $X := X - 1$

Prop. ① $P(D).\ \forall \sigma, \sigma', n.\ \sigma(X) = n \wedge \sigma(R) = m$

$D \Vdash \langle \sigma, c_{loop} \rangle \Downarrow \sigma' \Rightarrow$

$\sigma'(R) = m + \sum_{i=1}^{n} i.$

$\forall D.\ P(D).$

Proof.

By induction on derivation of D.

Case While-F

$\sigma = \sigma'$

$\langle \sigma, 0 < X \rangle \Downarrow$ false.

$\vdots$

Case While-T

$\langle \sigma, 0 < X \rangle \Downarrow$ true

$\langle \sigma, R := R + X;\ X := X - 1 \rangle \Downarrow \sigma_1$

$D_1 \Vdash \langle \sigma_1, c_{loop} \rangle \Downarrow \sigma'$

$\vdots$