Defn [Command Equivalence]

$$c_1 \sim c_2 \triangleq \forall \sigma, \sigma'.\ \langle \sigma, c_1 \rangle \Downarrow \sigma' \iff \langle \sigma, c_2 \rangle \Downarrow \sigma'.$$

- Reflexive: $c \sim c$.
- Symmetric: $c_1 \sim c_2 \Rightarrow c_2 \sim c_1$.
- Transitive: $c_1 \sim c_2 \wedge c_2 \sim c_3 \Rightarrow c_1 \sim c_3$.

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Prop $\underbrace{\boxed{\text{while } b \text{ do } c}}_{c_1} \sim \underbrace{\boxed{\text{if } b \text{ then } (c; \text{while } b \text{ do } c) \text{ else skip}}}_{c_2}$.

Proof. Let σ, σ' be stores.

We'll prove each direction of bi-implication.

($\Rightarrow$)

Assume $\langle \sigma, c_1 \rangle \Downarrow \sigma'$.

To show: $\langle \sigma, c_2 \rangle \Downarrow \sigma'$.

By inspection, last rule used in big step must have been While-T/F.

Case While-T:

$\langle \sigma, b \rangle \Downarrow \text{true}$

$\langle \sigma, c \rangle \Downarrow \sigma_1$

$\langle \sigma_1,$

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Prop $\underbrace{\boxed{\text{while } b \text{ do } c}}_{c_1} \sim \underbrace{\boxed{\text{if } b \text{ then } (c;\ \text{while } b \text{ do } c) \text{ else skip.}}}_{c_2}$

Proof. Let σ, σ' be stores.

We'll prove each direction of bi-implication.

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Assume $\langle \sigma, c_1 \rangle \Downarrow \sigma'$.

To show: $\langle \sigma, c_2 \rangle \Downarrow \sigma'$.

By inspection, last rule used in big step must have been While-T/F.

Case While-T:

$\langle \sigma, b \rangle \Downarrow \text{true}$

$\langle \sigma, c \rangle \Downarrow \sigma_1$

$\langle \sigma_1, \text{while } b \text{ do } c \rangle \Downarrow \sigma'$

Defn [Command Equivalence]

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Proof. Let σ, σ' be stores.

Case While-T:

$\langle \sigma, b \rangle \Downarrow \text{true}$

$\langle \sigma, c \rangle \Downarrow \sigma_1$

$\langle \sigma_1, c_1 \rangle \Downarrow \sigma'$

$$\dfrac{\dfrac{}{\langle \sigma, b \rangle \Downarrow \text{true}} \qquad \dfrac{\dfrac{}{\langle \sigma, c \rangle \Downarrow \sigma_1} \quad \dfrac{}{\langle \sigma_1, c_1 \rangle \Downarrow \sigma'}}{\langle \sigma, c; \text{while } b \text{ do } c \rangle \Downarrow \sigma'}\ \text{Seq}}{\langle \sigma, c_2 \rangle \Downarrow \sigma'}\ \text{If-T}$$

✓

Defn [Command Equivalence]

$$c_1 \sim c_2 \triangleq \boxed{\forall \sigma, \sigma'.\ \langle \sigma, c_1 \rangle \Downarrow \sigma' \Longleftrightarrow \langle \sigma, c_2 \rangle \Downarrow \sigma'}.$$

Prop $\underbrace{\boxed{\texttt{while } b \texttt{ do } c}}_{c_1} \sim \underbrace{\boxed{\texttt{if } b \texttt{ then } (c;\ \texttt{while } b \texttt{ do } c) \texttt{ else skip}}}_{c_2}$.

Proof. Let σ, σ' be stores.

$(\Longleftarrow)$ Assume $\langle \sigma, c_2 \rangle \Downarrow$

Defn [Command Equivalence]

$$c_1 \sim c_2 \triangleq \boxed{\forall \sigma, \sigma'.\ \langle \sigma, c_1 \rangle \Downarrow \sigma' \iff \langle \sigma, c_2 \rangle \Downarrow \sigma'}.$$

Prop $\underbrace{\texttt{while } b \texttt{ do } c}_{c_1} \sim \underbrace{\texttt{if } b \texttt{ then } (c;\ \texttt{while } b \texttt{ do } c) \texttt{ else skip}}_{c_2}$.

Proof. Let σ, σ' be stores.

Case While-F:

$\langle \sigma, b \rangle \Downarrow \text{false}$

$\sigma = \sigma'$

$$\dfrac{\langle \sigma, b \rangle \Downarrow \text{false} \qquad \dfrac{}{\langle \sigma, \texttt{skip} \rangle \Downarrow \sigma}\ \text{Skip}}{\langle \sigma, c_2 \rangle \Downarrow \sigma}\ \text{If-F}$$

✓✓

Defn [Command Equivalence]

$$c_1 \sim c_2 \triangleq \boxed{\forall \sigma, \sigma'.\ \langle \sigma, c_1 \rangle \Downarrow \sigma' \iff \langle \sigma, c_2 \rangle \Downarrow \sigma'}.$$

Prop $\underbrace{\text{while } b \text{ do } c}_{c_1} \sim \underbrace{\text{if } b \text{ then } (c; \text{while } b \text{ do } c) \text{ else skip}}_{c_2}$.

Proof. Let σ, σ' be stores.

($\Longleftarrow$) Assume $\langle \sigma, c_2 \rangle \Downarrow \sigma'$.

By inspection, last rule used was If-T/F.

Case If-T:

$\langle \sigma, b \rangle \Downarrow \text{true}$

$\langle \sigma, c; c_1 \rangle \Downarrow \sigma'$

By inspection, must have used Seq rule.

$\langle \sigma, c \rangle \Downarrow \sigma_1$

$\langle \sigma_1, c_1 \rangle \Downarrow \sigma'$

$$\frac{\langle \sigma, b \rangle \Downarrow \text{true} \quad \langle \sigma, c \rangle \Downarrow \sigma_1 \quad \overline{\langle \sigma_1, c_1 \rangle \Downarrow \sigma'}}{\langle \sigma, c_1 \rangle \Downarrow \sigma'}$$

Defn [Command Equivalence]

$$c_1 \sim c_2 \triangleq \boxed{\forall \sigma, \sigma'.\ \langle \sigma, c_1 \rangle \Downarrow \sigma' \iff \langle \sigma, c_2 \rangle \Downarrow \sigma'}.$$

if b then (c ; while b do c) else skip.

$c ; \text{skip} \sim \text{skip} ; c \sim c$

$(c_1 ; c_2) ; c_3 \sim c_1 ; (c_2 ; c_3)$

$x := a_1 ; x := a_2 \sim x := a_2$

$x := a ; y := b \sim y := b ; x := a$

– (assumes $x \neq y$)

Case If-T:

$\langle \sigma, b \rangle \Downarrow \text{true}$

$\langle \sigma, c ; c_1 \rangle \Downarrow \sigma'$

inspection, must have used Seq Rule.

$\langle \sigma, c \rangle \Downarrow \sigma_1$

$\langle \sigma_1, c_1 \rangle \Downarrow \sigma'$