Lecture 28
Propositions as Types
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Logics = Type Systems
We have used inference rules to build up inductively defined sets of PL concepts: operational steps, valid Hoare triples, associations between terms and types, etc.

Logicians use the same kind of notation to build up the set of true logical formulas.
Inference Rules for Logic

We have used inference rules to build up inductively defined sets of PL concepts: operational steps, valid Hoare triples, associations between terms and types, etc.

Logicians use the same kind of notation to build up the set of true logical formulas.

Here’s a rule from natural deduction, a constructive logic invented by logician Gerhard Gentzen in 1935:

\[
\frac{\phi \quad \psi}{\phi \land \psi} \quad \land\text{-INTRO}
\]

Given a proof of $\phi$ and a proof of $\psi$, the rule lets you construct a proof of $\phi \land \psi$. 
Natural Deduction

Let’s use our usual 4110 tools to define the set of true formulas (“theorems”).
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We’ll start with a grammar for formulas:

\[
\phi ::= \top \quad \bot \quad X \quad \phi \land \psi \quad \phi \lor \psi \quad \phi \rightarrow \psi \quad \neg \phi \quad \forall X. \phi
\]

where \( X \) ranges over Boolean variables and \( \neg \phi \) is an abbreviation for \( \phi \rightarrow \bot \).
Natural Deduction

Let’s define a judgment that a formula is true given a set of assumptions $\Gamma$:

$$\Gamma \vdash \phi$$

where $\Gamma$ is just a list of formulas.
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**Examples:**
- $\vdash A \land B \rightarrow A$
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**Examples:**

- $\vdash A \land B \rightarrow A$
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**Examples:**

- $\vdash A \land B \rightarrow A$
- $\vdash \neg(A \land B) \rightarrow \neg A \lor \neg B$
- $A, B, C \vdash B$
Let’s write the rules for our judgment:

\[
\Gamma \vdash \phi \quad \Gamma \vdash \psi \\
\hline
\Gamma \vdash \phi \land \psi
\]  
\text{\land\text{-}\text{INTRO}}
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\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi} \quad \land\text{-INTRO}
\]

\[
\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \phi} \quad \land\text{-ELIM1}
\]

\[
\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \psi} \quad \land\text{-ELIM2}
\]
Natural Deduction

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\[
\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \rightarrow\text{-INTRO}
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Let’s write the rules for our judgment:

\[
\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi} \quad \text{\neg\text{-INTRO}}
\]

\[
\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \phi} \quad \text{\neg\text{-ELIM1}}
\]

\[
\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \psi} \quad \text{\neg\text{-ELIM2}}
\]

\[
\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \text{\neg\text{-INTRO}}
\]

...and so on.
Natural Deduction

**Axiom**

\[ \Gamma, \phi \vdash \phi \]

**→-Intro**

\[ \Gamma, \phi \vdash \psi \quad \Gamma \vdash \phi \quad \Gamma \vdash \phi \rightarrow \psi \]

**→-Elim**

\[ \Gamma \vdash \phi \rightarrow \psi \quad \Gamma \vdash \phi \quad \Gamma \vdash \psi \]

**∧-Intro**

\[ \Gamma \vdash \phi \quad \Gamma \vdash \psi \quad \Gamma \vdash \phi \land \psi \]

**∧-Elim1**

\[ \Gamma \vdash \phi \land \psi \quad \Gamma \vdash \phi \]

**∧-Elim2**

\[ \Gamma \vdash \phi \land \psi \quad \Gamma \vdash \psi \]

**∨-Intro1**

\[ \Gamma \vdash \phi \quad \Gamma \vdash \phi \rightarrow \chi \quad \Gamma \vdash \phi \lor \psi \]

**∨-Intro2**

\[ \Gamma \vdash \psi \quad \Gamma \vdash \phi \lor \psi \]

**∨-Elim**

\[ \Gamma \vdash \phi \lor \psi \quad \Gamma \vdash \phi \rightarrow \chi \quad \Gamma \vdash \psi \rightarrow \chi \quad \Gamma \vdash \chi \]

**∀-Intro**

\[ \Gamma, P \vdash \phi \quad \Gamma \vdash \forall P. \phi \]

**∀-Elim**

\[ \Gamma \vdash \forall P. \phi \quad \Gamma \vdash \phi \{\psi/P\} \]
Natural Deduction

Let’s try a proof! We can write a proof that $A \land B \rightarrow B \land A$ is a theorem.
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\[
\begin{align*}
A \land B \vdash A \land B \quad \text{AXIOM} \\
A \land B \vdash B \quad \text{-ELIM2} \\
A \land B \vdash A \quad \text{AXIOM} \\
A \land B \vdash B \quad \text{-ELIM1} \\
A \land B \vdash B \land A \\
\vdash A \land B \rightarrow B \land A \quad \rightarrow\text{-INTRO}
\end{align*}
\]

Does this look familiar?
Let’s try a proof! We can write a proof that $A \land B \rightarrow B \land A$ is a theorem.

\[
\begin{align*}
A \land B & \vdash A \land B \\
\text{AXIOM} \\
A \land B & \vdash A \land B \\
\text{AXIOM} \\
A \land B & \vdash A \\
\land\text{-ELIM2} \\
A \land B & \vdash B \\
\land\text{-ELIM1} \\
A \land B & \vdash B \land A \\
\land\text{-INTRO} \\
A \land B & \vdash B \land A \\
\text{→-INTRO} \\
\end{align*}
\]

Does this look familiar?

\[
\begin{align*}
x: A \times B & \vdash x: A \times B \\
\text{T-_VAR} \\
x: A \times B & \vdash \#2 x: B \\
\text{T-#1} \\
\end{align*}
\]

\[
\begin{align*}
x: A \times B & \vdash x: A \times B \\
\text{T-_VAR} \\
x: A \times B & \vdash \#1 x: A \\
\text{T-#2} \\
\end{align*}
\]

\[
\begin{align*}
x: A \times B & \vdash (\#2 x, \#1 x): B \times A \\
\text{T-PAIR} \\
\end{align*}
\]

\[
\begin{align*}
\vdash \lambda x. (\#2 x, \#1 x): A \times B \rightarrow B \times A \\
\text{T-ABS} \\
\end{align*}
\]
Propositions as Types

Every natural deduction proof tree has a corresponding type tree in System F with product and sum types! And vice-versa!

<table>
<thead>
<tr>
<th>Type Systems</th>
<th>Formal Logic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$ Type</td>
<td>$\phi$ Formula</td>
</tr>
<tr>
<td>$\tau$ is inhabited</td>
<td>$\phi$ is a theorem</td>
</tr>
<tr>
<td>$e$ Well-typed expression</td>
<td>$\pi$ Proof</td>
</tr>
</tbody>
</table>

A program with a given type acts as a witness that the type’s corresponding formula is true.
Propositions as Types

Every type rule in System F with product and sum types corresponds 1-1 with a proof rule in natural deduction:

<table>
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<tr>
<td>$\to$</td>
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</tr>
<tr>
<td>$\times$</td>
<td>$\land$</td>
</tr>
<tr>
<td>$+$</td>
<td>$\lor$</td>
</tr>
<tr>
<td>$\forall$</td>
<td>$\forall$</td>
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You can even add existential types to correspond to existential quantification. It still works!
Propositions as Types

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<tr>
<td>→ Function</td>
<td>→ Implication</td>
</tr>
<tr>
<td>× Product</td>
<td>∧ Conjunction</td>
</tr>
<tr>
<td>+ Sum</td>
<td>∨ Disjunction</td>
</tr>
<tr>
<td>∀ Universal</td>
<td>∀ Quantifier</td>
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Is this a coincidence? Natural deduction was invented by a German logician in 1935. Types for the $\lambda$-calculus were invented by Church at Princeton in 1940.
<table>
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<th>Propositions as Types Through the Ages</th>
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<tr>
<td><strong>Natural Deduction</strong></td>
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<tr>
<td>Gentzen (1935)</td>
</tr>
<tr>
<td><strong>Type Schemes</strong></td>
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<tr>
<td><strong>System F</strong></td>
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<tr>
<td><strong>Modal Logic</strong></td>
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<tr>
<td>Lewis (1910)</td>
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<tr>
<td><strong>Classical–Intuitionistic Embedding</strong></td>
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<td>Gödel (1933)</td>
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Term Assignment

This all means that we have a new way of proving theorems: writing programs!
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To prove a formula $\phi$:

1. Convert the $\phi$ into its corresponding type $\tau$.
2. Find some program $v$ that has the type $\tau$.
3. Realize that the existence of $v$ implies a type tree for $\vdash v : \tau$, which implies a proof tree for $\vdash \phi$. 
Negation and Continuations

Let’s explore one extension. We’d like to use this rule from classical logic:

\[
\Gamma \vdash \phi \\
\hline
\Gamma \vdash \neg\neg \phi
\]

but there’s no obvious correspondence in System F.

\[
\neg \phi \text{ is shorthand for } \phi \rightarrow \bot
\]

So \(\neg\neg \phi\) corresponds to the System F function type \((\tau \rightarrow \bot) \rightarrow \bot\).

What we need is a way to take any program of any type \(\tau\) and turn it into a program of type \((\tau \rightarrow \bot) \rightarrow \bot\).

Shockingly, that’s exactly what the CPS transform does! A \(\tau\) becomes a function that takes a continuation \(\tau \rightarrow \bot\) and invokes it, producing \(\bot\).
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