# CS 4110

# Programming Languages & Logics

## Lecture 25
## Type Inference

# Review: Polymorphic $\lambda$-Calculus

## Syntax

$$e ::= n \mid x \mid \lambda x\!:\!\tau.\, e \mid e_1\, e_2 \mid \Lambda X.\, e \mid e\,[\tau]$$

$$v ::= n \mid \lambda x\!:\!\tau.\, e \mid \Lambda X.\, e$$

## Dynamic Semantics

$$E ::= [\cdot] \mid E\, e \mid v\, E \mid E\,[\tau]$$

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']} \qquad \frac{}{(\lambda x\!:\!\tau.\, e)\, v \rightarrow e\{v/x\}} \qquad \frac{}{(\Lambda X.\, e)\,[\tau] \rightarrow e\{\tau/X\}}$$

# Review: Polymorphic $\lambda$-Calculus

$$\frac{}{\Delta, \Gamma \vdash n : \mathbf{int}} \qquad \frac{\Gamma(x) = \tau}{\Delta, \Gamma \vdash x : \tau}$$

$$\frac{\Delta, \Gamma, x : \tau \vdash e : \tau' \quad \Delta \vdash \tau \text{ ok}}{\Delta, \Gamma \vdash \lambda x : \tau.\, e : \tau \to \tau'} \qquad \frac{\Delta, \Gamma \vdash e_1 : \tau \to \tau' \quad \Delta, \Gamma \vdash e_2 : \tau}{\Delta, \Gamma \vdash e_1\, e_2 : \tau'}$$

$$\frac{\Delta \cup \{X\}, \Gamma \vdash e : \tau}{\Delta, \Gamma \vdash \Lambda X.\, e : \forall X.\, \tau} \qquad \frac{\Delta, \Gamma \vdash e : \forall X.\, \tau' \quad \Delta \vdash \tau \text{ ok}}{\Delta, \Gamma \vdash e\, [\tau] : \tau'\{\tau / X\}}$$

# Review: Polymorphic $\lambda$-Calculus

Polymorphism let us write a doubling function that works for *any* type of function:

$$\text{double} \triangleq \Lambda X.\, \lambda f{:}X \to X.\, \lambda x{:}X.\, f\,(f\,x).$$

The type of this expression is:

$$\forall X.\, (X \to X) \to X \to X$$

You can use the polymorphic function by providing a type:

$$\text{double}\,[\mathbf{int}]\,(\lambda n{:}\mathbf{int}.\, n + 1)\; 7$$

# Type Inference

In languages like OCaml, programmers don't have to annotate their programs with $\forall X.\ \tau$ or $e\ [\tau]$.

# Type Inference

In languages like OCaml, programmers don't have to annotate their programs with $\forall X.\, \tau$ or $e\ [\tau]$.

For example, we can write:

```
let double f x = f (f x)
```

and OCaml will figure out that the type is:

```
('a → 'a) → 'a → 'a
```

which is equivalent to the same System F type:

$$\forall A.\, (A \to A) \to A \to A$$

# Type Inference

In languages like OCaml, programmers don't have to annotate their programs with $\forall X.\ \tau$ or $e\ [\tau]$.

We can also write

```
double (fun x → x+1) 7
```

and OCaml will infer that the polymorphic function `double` is instantiated at the type `int`.

# ML Polymorphism

However, polymorphism in OCaml and other MLs, has some restrictions to ensure that type inference remains *decidable*.

# ML Polymorphism

However, polymorphism in OCaml and other MLs, has some restrictions to ensure that type inference remains *decidable*.

These restrictions, called *prenex polymorphism*, stipulate that $\forall$s may only appear in the “outermost” position.

# ML Polymorphism

However, polymorphism in OCaml and other MLs, has some restrictions to ensure that type inference remains *decidable*.

These restrictions, called *prenex polymorphism*, stipulate that $\forall$s may only appear in the "outermost" position.

## Examples

- Prenex: $\forall \alpha.\, \alpha \to \alpha$

# ML Polymorphism

However, polymorphism in OCaml and other MLs, has some restrictions to ensure that type inference remains *decidable*.

These restrictions, called *prenex polymorphism*, stipulate that $\forall$s may only appear in the "outermost" position.

## Examples

- Prenex: $\forall \alpha.\, \alpha \to \alpha$
- Not prenex: $(\forall \alpha.\, \alpha \to \alpha) \to \textbf{int}$

# ML Polymorphism

However, polymorphism in OCaml and other MLs, has some restrictions to ensure that type inference remains *decidable*.

These restrictions, called *prenex polymorphism*, stipulate that $\forall$s may only appear in the "outermost" position.

## Examples

- Prenex: $\forall\alpha.\, \alpha \rightarrow \alpha$
- Not prenex: $(\forall\alpha.\, \alpha \rightarrow \alpha) \rightarrow \textbf{int}$

These restrictions have the following practical ramifications:

- Can't instantiate type variables with polymorphic types
- Can't put a polymorphic type on the left of an arrow

# Example

These restrictions mean that certain terms that are typeable in System F are not typeable in ML!

# Example

These restrictions mean that certain terms that are typeable in System F are not typeable in ML!

```
        OCaml version 4.01.0

# fun x -> x x;;
Error: This expression has type 'a -> 'b
    but an expression was expected of type 'a
    The type variable 'a occurs inside 'a -> 'b
```

# Type Inference

Type inference may be undecidable for the polymorphic $\lambda$-calculus and OCaml, but it is possible for the simply-tpyed $\lambda$-calculus!

# Type Inference

Type inference may be undecidable for the polymorphic $\lambda$-calculus and OCaml, but it is possible for the simply-tpyed $\lambda$-calculus!

Type inference for the STLC means guessing a $\tau$ in every abstraction in an *untyped* version:

$$\lambda x.\, e$$

to produce a *typed* program:

$$\lambda x{:}\tau.\, e$$

that we can use in the typing rule for functions.

# Example

Here's an untyped program:

$\lambda a.\ \lambda b.\ \lambda c.\ \text{if } a\ (b + 1) \text{ then } b \text{ else } c$

# Example

Here's an untyped program:

$\lambda a.\, \lambda b.\, \lambda c.\ \text{if } a\ (b+1) \text{ then } b \text{ else } c$

Informal inference:

# Example

Here's an untyped program:

$\lambda a.\, \lambda b.\, \lambda c.\, \text{if } a\ (b + 1) \text{ then } b \text{ else } c$

Informal inference:

- $b$ must be **int**

# Example

Here's an untyped program:

$\lambda a.\, \lambda b.\, \lambda c.$ if $a\ (b+1)$ then $b$ else $c$

Informal inference:

- $b$ must be **int**
- $a$ must be some kind of function

# Example

Here's an untyped program:

$\lambda a.\, \lambda b.\, \lambda c.\, \text{if } a\ (b + 1) \text{ then } b \text{ else } c$

Informal inference:

- $b$ must be **int**
- $a$ must be some kind of function
- the argument type of $a$ must be the same as $b + 1$

# Example

Here's an untyped program:

$\lambda a.\ \lambda b.\ \lambda c.\ \text{if } a\ (b + 1) \text{ then } b \text{ else } c$

Informal inference:

- $b$ must be **int**
- $a$ must be some kind of function
- the argument type of $a$ must be the same as $b + 1$
- the result type of $a$ must be **bool**

# Example

Here's an untyped program:

$\lambda a.\, \lambda b.\, \lambda c.\, \text{if } a\ (b + 1) \text{ then } b \text{ else } c$

Informal inference:

- *b* must be **int**
- *a* must be some kind of function
- the argument type of *a* must be the same as $b + 1$
- the result type of *a* must be **bool**
- the type of *c* must be the same as *b*

# Example

Here's an untyped program:

$\lambda a.\ \lambda b.\ \lambda c.\ \text{if}\ a\ (b + 1)\ \text{then}\ b\ \text{else}\ c$

Informal inference:

- $b$ must be **int**
- $a$ must be some kind of function
- the argument type of $a$ must be the same as $b + 1$
- the result type of $a$ must be **bool**
- the type of $c$ must be the same as $b$

Putting all these pieces together:

$\lambda a : \textbf{int} \to \textbf{bool}.\ \lambda b : \textbf{int}.\ \lambda c : \textbf{int}.\ \text{if}\ a\ (b + 1)\ \text{then}\ b\ \text{else}\ c$

# Constraint-Based Inference

Let’s automate type inference!

# Constriant-Based Inference

Let's automate type inference!

We introduce a new judgment:

$$\Gamma \vdash e : \tau \mid C$$

Given a typing context $\Gamma$ and an expression $e$, it generates a set of *constraints*—equations between types.

# Constriant-Based Inference

Let's automate type inference!

We introduce a new judgment:

$$\Gamma \vdash e : \tau \mid C$$

Given a typing context Γ and an expression *e*, it generates a set of *constraints*—equations between types.

If these constraints are solvable, then *e* can be well-typed in Γ.

A solution to a set of constraints is a *type substitution* $\sigma$ that, for each equation, makes both sides syntactically equal.

# STLC for Type Inference

Let's define the type inference judgment for this STLC language:

$$e ::= x \mid \lambda x\!:\!\tau.\, e \mid e_1\, e_2 \mid n \mid e_1 + e_2$$
$$\tau ::= \textbf{int} \mid X \mid \tau_1 \to \tau_2$$

You can use a type variable $X$ wherever you want to have a type inferred.

# Constraint-Based Typing Judgment

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau \mid \emptyset} \text{ CT-VAR}$$

# Constraint-Based Typing Judgment

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau \mid \emptyset} \ \textsc{CT-Var} \qquad\qquad \frac{}{\Gamma \vdash n : \textbf{int} \mid \emptyset} \ \textsc{CT-Int}$$

# Constraint-Based Typing Judgment

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau \mid \emptyset}\ \textsc{CT-Var} \qquad\qquad \frac{}{\Gamma \vdash n : \mathbf{int} \mid \emptyset}\ \textsc{CT-Int}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \mid C_1 \quad \Gamma \vdash e_2 : \tau_2 \mid C_2}{\Gamma \vdash e_1 + e_2 : \mathbf{int} \mid C_1 \cup C_2 \cup \{\tau_1 = \mathbf{int}, \tau_2 = \mathbf{int}\}}\ \textsc{CT-Add}$$

# Constraint-Based Typing Judgment

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau \mid \emptyset}\ \text{CT-Var} \qquad\qquad \frac{}{\Gamma \vdash n : \mathbf{int} \mid \emptyset}\ \text{CT-Int}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \mid C_1 \quad \Gamma \vdash e_2 : \tau_2 \mid C_2}{\Gamma \vdash e_1 + e_2 : \mathbf{int} \mid C_1 \cup C_2 \cup \{\tau_1 = \mathbf{int}, \tau_2 = \mathbf{int}\}}\ \text{CT-Add}$$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \mid C}{\Gamma \vdash \lambda x : \tau_1 .\, e : \tau_1 \to \tau_2 \mid C}\ \text{CT-Abs}$$

# Constraint-Based Typing Judgment

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau \mid \emptyset} \text{ CT-Var} \qquad \frac{}{\Gamma \vdash n : \textbf{int} \mid \emptyset} \text{ CT-Int}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \mid C_1 \quad \Gamma \vdash e_2 : \tau_2 \mid C_2}{\Gamma \vdash e_1 + e_2 : \textbf{int} \mid C_1 \cup C_2 \cup \{\tau_1 = \textbf{int}, \tau_2 = \textbf{int}\}} \text{ CT-Add}$$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \mid C}{\Gamma \vdash \lambda x : \tau_1 . \, e : \tau_1 \to \tau_2 \mid C} \text{ CT-Abs}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \tau_1 \mid C_1 \quad \Gamma \vdash e_2 : \tau_2 \mid C_2 \\ X \text{ fresh} \quad C' = C_1 \cup C_2 \cup \{\tau_1 = \tau_2 \to X\}\end{array}}{\Gamma \vdash e_1 \, e_2 : X \mid C'} \text{ CT-App}$$

# Solving Constraints

A *type substitution* is a finite map from type variables to types.

Example: The substitution

$$[X \mapsto \textbf{int}, Y \mapsto \textbf{int} \to \textbf{int}]$$

maps type variable $X$ to **int** and $Y$ to $\textbf{int} \to \textbf{int}$.

# Type Substitution

We can define substitution of type variables formally:

# Type Substitution

We can define substitution of type variables formally:

$$\sigma(X) \triangleq \begin{cases} \tau & \text{if } X \mapsto \tau \in \sigma \\ X & \text{if } X \text{ not in the domain of } \sigma \end{cases}$$

# Type Substitution

We can define substitution of type variables formally:

$$\sigma(X) \triangleq \begin{cases} \tau & \text{if } X \mapsto \tau \in \sigma \\ X & \text{if } X \text{ not in the domain of } \sigma \end{cases}$$

$$\sigma(\mathbf{int}) \triangleq \mathbf{int}$$

# Type Substitution

We can define substitution of type variables formally:

$$\sigma(X) \triangleq \begin{cases} \tau & \text{if } X \mapsto \tau \in \sigma \\ X & \text{if } X \text{ not in the domain of } \sigma \end{cases}$$

$$\sigma(\mathbf{int}) \triangleq \mathbf{int}$$

$$\sigma(\tau \to \tau') \triangleq \sigma(\tau) \to \sigma(\tau')$$

# Type Substitution

We can define substitution of type variables formally:

$$\sigma(X) \triangleq \begin{cases} \tau & \text{if } X \mapsto \tau \in \sigma \\ X & \text{if } X \text{ not in the domain of } \sigma \end{cases}$$

$$\sigma(\mathbf{int}) \triangleq \mathbf{int}$$

$$\sigma(\tau \to \tau') \triangleq \sigma(\tau) \to \sigma(\tau')$$

We don't need to worry about avoiding variable capture: all type variables are "free."

# Type Substitution

We can define substitution of type variables formally:

$$\sigma(X) \triangleq \begin{cases} \tau & \text{if } X \mapsto \tau \in \sigma \\ X & \text{if } X \text{ not in the domain of } \sigma \end{cases}$$

$$\sigma(\mathbf{int}) \triangleq \mathbf{int}$$

$$\sigma(\tau \to \tau') \triangleq \sigma(\tau) \to \sigma(\tau')$$

We don't need to worry about avoiding variable capture: all type variables are "free."

Given two substitutions $\sigma_1$ and $\sigma_2$, we write $\sigma_1 \circ \sigma_2$ for their composition: $(\sigma_1 \circ \sigma_2)(\tau) = \sigma_1(\sigma_2(\tau))$.

# Unification

Our constraints are of the form $\tau = \tau'$.

# Unification

Our constraints are of the form $\tau = \tau'$.

We say that a substitution $\sigma$ *unifies* constraint $\tau = \tau'$ if $\sigma(\tau) = \sigma(\tau')$.

We say that substitution $\sigma$ *satisfies* (or *unifies*) set of constraints *C* if $\sigma$ unifies every constraint in *C*.

# Unification

If:

- $\Gamma \vdash e : \tau \mid C$, and
- $\sigma$ satisfies $C$,

then $e$ has type $\tau'$ under $\Gamma$,
where $\sigma(\tau) = \tau'$.

If there are no substitutions that satisfy $C$, then $e$ is not typeable.

# Unification

If:

- $\Gamma \vdash e : \tau \mid C$, and
- $\sigma$ satisfies $C$,

then $e$ has type $\tau'$ under $\Gamma$,
where $\sigma(\tau) = \tau'$.

If there are no substitutions that satisfy $C$, then $e$ is not typeable.

So let's find a substitution $\sigma$ that unifies a set of constraints $C$!

# Unification Algorithm

# Unification Algorithm

$\mathit{unify}(\emptyset) \triangleq []$ (the empty substitution)

# Unification Algorithm

*unify*$(\emptyset) \triangleq []$ (the empty substitution)

*unify*$(\{\tau = \tau'\} \cup C') \triangleq$
if $\tau = \tau'$ then
    *unify*$(C')$

# Unification Algorithm

$\mathit{unify}(\emptyset) \triangleq []$ (the empty substitution)

$\mathit{unify}(\{\tau = \tau'\} \cup C') \triangleq$
if $\tau = \tau'$ then
$\quad \mathit{unify}(C')$
else if $\tau = X$ and $X$ not a free variable of $\tau'$ then
$\quad \mathit{unify}(C'\{\tau'/X\}) \circ [X \mapsto \tau']$

# Unification Algorithm

$\mathit{unify}(\emptyset) \triangleq []$ (the empty substitution)

$\mathit{unify}(\{\tau = \tau'\} \cup C') \triangleq$
if $\tau = \tau'$ then
$\quad \mathit{unify}(C')$
else if $\tau = X$ and $X$ not a free variable of $\tau'$ then
$\quad \mathit{unify}(C'\{\tau'/X\}) \circ [X \mapsto \tau']$
else if $\tau' = X$ and $X$ not a free variable of $\tau$ then
$\quad \mathit{unify}(C'\{\tau/X\}) \circ [X \mapsto \tau]$

# Unification Algorithm

$unify(\emptyset) \triangleq []$ (the empty substitution)

$unify(\{\tau = \tau'\} \cup C') \triangleq$
if $\tau = \tau'$ then
  $unify(C')$
else if $\tau = X$ and $X$ not a free variable of $\tau'$ then
  $unify(C'\{\tau'/X\}) \circ [X \mapsto \tau']$
else if $\tau' = X$ and $X$ not a free variable of $\tau$ then
  $unify(C'\{\tau/X\}) \circ [X \mapsto \tau]$
else if $\tau = \tau_o \to \tau_1$ and $\tau' = \tau'_o \to \tau'_1$ then
  $unify(C' \cup \{\tau_0 = \tau'_0, \tau_1 = \tau'_1\})$

# Unification Algorithm

$unify(\emptyset) \triangleq []$ (the empty substitution)

$unify(\{\tau = \tau'\} \cup C') \triangleq$
if $\tau = \tau'$ then
  $unify(C')$
else if $\tau = X$ and $X$ not a free variable of $\tau'$ then
  $unify(C'\{\tau'/X\}) \circ [X \mapsto \tau']$
else if $\tau' = X$ and $X$ not a free variable of $\tau$ then
  $unify(C'\{\tau/X\}) \circ [X \mapsto \tau]$
else if $\tau = \tau_o \rightarrow \tau_1$ and $\tau' = \tau'_o \rightarrow \tau'_1$ then
  $unify(C' \cup \{\tau_0 = \tau'_0, \tau_1 = \tau'_1\})$
else
  *fail*

# Unification Properties

The unification algorithm always terminates.

# Unification Properties

The unification algorithm always terminates.

The solution, if it exists, is the most general solution: if $\sigma = \mathit{unify}(C)$ and $\sigma'$ is a solution to $C$, then there is some $\sigma''$ such that $\sigma' = (\sigma'' \circ \sigma)$.