Review: $\lambda$-calculus

Syntax

\[
\begin{align*}
  e & ::= x \mid e_1 e_2 \mid \lambda x. e \\
  \nu & ::= \lambda x. e
\end{align*}
\]

Semantics

\[
\begin{align*}
  e_1 & \rightarrow e'_1 \\
  e_1 e_2 & \rightarrow e'_1 e_2 \\
  e & \rightarrow e' \\
  \nu e & \rightarrow \nu e'
\end{align*}
\]

\[
(\lambda x. e) \nu \rightarrow e\{\nu/x\}^\beta
\]
Rewind: Currying

This is just a function that returns a function:

\[ \text{ADD} \triangleq \lambda x. \lambda y. x + y \]

\[ \text{ADD } 38 \rightarrow \lambda y. 38 + y \]

\[ \text{ADD } 38 \ 4 = (\text{ADD } 38) \ 4 \rightarrow 42 \]

**Informally**, you can think of it as a *curried* function that takes two arguments, one after the other.

But that’s just a way to get intuition. The \( \lambda \)-calculus only has one-argument functions.
Here are the syntax and CBV semantics of $\lambda$-calculus:

$$
e ::= x \mid \lambda x. e \mid e_1 e_2 \\
\nu ::= \lambda x. e$$

$$
e_1 \rightarrow e'_1 \\
\frac{e_1 e_2 \rightarrow e'_1 e_2}{e_1 e_2 \rightarrow e'_1 e_2}
$$

$$
e \rightarrow e' \\
\frac{e \rightarrow e'}{\nu e \rightarrow \nu e'}
$$

$$
(\lambda x. e) \nu \rightarrow e\{\nu/x\} \ \beta
$$

There are two kinds of rules: congruence rules that specify evaluation order and computation rules that specify the “interesting” reductions.
Evaluation contexts let us separate out these two kinds of rules.
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An evaluation context $E$ is an expression with a “hole” in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= [\cdot] \mid E \, e \mid \nu \, E$$
Evaluation contexts let us separate out these two kinds of rules.

An evaluation context $E$ is an expression with a “hole” in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= [\cdot] \mid E \ e \mid \nu \ E$$

We write $E[e]$ to mean the evaluation context $E$ where the hole has been replaced with the expression $e$. 
Examples

\[ E_1 = [\cdot] (\lambda x. x) \]
\[ E_1[\lambda y. y y] = (\lambda y. y y) \lambda x. x \]
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\[ E_1 = [\cdot] (\lambda x. x) \]
\[ E_1[\lambda y. y y] = (\lambda y. y y) \lambda x. x \]

\[ E_2 = (\lambda z. z z) [\cdot] \]
\[ E_2[\lambda x. \lambda y. x] = (\lambda z. z z) (\lambda x. \lambda y. x) \]
Examples

\[ E_1 = [\cdot] (\lambda x. x) \]

\[ E_1[\lambda y. y y] = (\lambda y. y y) \lambda x. x \]

\[ E_2 = (\lambda z. z z) [\cdot] \]

\[ E_2[\lambda x. \lambda y. x] = (\lambda z. z z) (\lambda x. \lambda y. x) \]

\[ E_3 = ([\cdot] \lambda x. x x) ((\lambda y. y) (\lambda y. y)) \]

\[ E_3[\lambda f. \lambda g. f g] = ((\lambda f. \lambda g. f g) \lambda x. x x) ((\lambda y. y) (\lambda y. y)) \]
With evaluation contexts, we can define the evaluation semantics for the CBV $\lambda$-calculus with just two rules: one for evaluation contexts, and one for $\beta$-reduction.

With this syntax:

$E ::= \mathbf{[e]} | Ee | vE$

The small-step rules are:

$e \rightarrow e'$

$E\mathbf{[e]} \rightarrow E\mathbf{[e']}$

$(\lambda x. e) v \rightarrow e\{v/\ x\}$
With evaluation contexts, we can define the evaluation semantics for the CBV $\lambda$-calculus with just two rules: one for evaluation contexts, and one for $\beta$-reduction.

With this syntax:

$$E ::= [] | E \ e | v \ E$$

The small-step rules are:

$$e \to e'$$

$$E[e] \to E[e']$$

$$(\lambda x. e) \ v \to e\{v/x\} \ \beta$$
We can also define the semantics of CBN $\lambda$-calculus with evaluation contexts.
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For call-by-name, the syntax for evaluation contexts is different:

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$$E ::= [\cdot] \mid E\ e$$

But the small-step rules are the same:

$$e \rightarrow e' \quad \frac{E[e] \rightarrow E[e']}{(\lambda x. e)\ e' \rightarrow e\ {e' / x}}_{\beta}$$
The pure $\lambda$-calculus contains only functions as values. It is not exactly easy to write large or interesting programs in the pure $\lambda$-calculus. We can however encode objects, such as booleans, and integers.
Booleans

We need to define functions TRUE, FALSE, AND, NOT, IF, and other operators that behave as follows:

\[
\begin{align*}
\text{AND TRUE FALSE} &= \text{FALSE} \\
\text{NOT FALSE} &= \text{TRUE} \\
\text{IF TRUE } e_1 e_2 &= e_1 \\
\text{IF FALSE } e_1 e_2 &= e_2
\end{align*}
\]

Let's start by defining TRUE and FALSE:

\[
\text{TRUE } ≜ \lambda x. \lambda y. x \\
\text{FALSE } ≜ \lambda x. \lambda y. y
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We want the function IF to behave like

$$\lambda b. \lambda t. \lambda f. \text{if } b \text{ is our term TRUE then } t, \text{ otherwise } f$$
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We can rely on the way we defined TRUE and FALSE:

\[ \text{IF } \triangleq \lambda b. \lambda t. \lambda f. b \, t \, f \]
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\[ \text{IF} \triangleq \lambda b. \lambda t. \lambda f. b \, t \, f \]

We can also write the standard Boolean operators.

\[ \text{NOT} \triangleq \]

\[ \text{AND} \triangleq \]

\[ \text{OR} \triangleq \]
Booleans

We want the function IF to behave like

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\[ \text{IF} \triangleq \lambda b. \lambda t. \lambda f. b \; t \; f \]

We can also write the standard Boolean operators.

\[ \text{NOT} \triangleq \lambda b. \; b \; \text{FALSE} \; \text{TRUE} \]

\[ \text{AND} \triangleq \lambda b_1. \lambda b_2. \; b_1 \; b_2 \; \text{FALSE} \]

\[ \text{OR} \triangleq \lambda b_1. \lambda b_2. \; b_1 \; \text{TRUE} \; b_2 \]
Let’s encode the natural numbers!

We’ll write $\bar{n}$ for the encoding of the number $n$. The central function we’ll need is a *successor* operation:

$$\text{SUCC } \bar{n} = \bar{n} + 1$$
Church Numerals

Church numerals encode a number $n$ as a function that takes $f$ and $x$, and applies $f$ to $x$ $n$ times.

\[
\begin{align*}
\bar{0} & \triangleq \lambda f. \lambda x. x \\
\bar{1} & \triangleq \lambda f. \lambda x. fx \\
\bar{2} & \triangleq \lambda f. \lambda x. f(fx)
\end{align*}
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\bar 1 & \triangleq \lambda f. \lambda x. fx \\
\bar 2 & \triangleq \lambda f. \lambda x. f(fx)
\end{align*}
\]

We can write a successor function that “inserts” another application of \( f \):

\[
\text{SUCC} \triangleq \lambda n. \lambda f. \lambda x. f(nfx)
\]
Given the definition of SUCC, we can define addition. Intuitively, the natural number $n_1 + n_2$ is the result of applying the successor function $n_1$ times to $n_2$.

\[ \text{PLUS} \triangleq \lambda n_1. \lambda n_2. n_1 \text{SUCC} n_2 \]
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\text{PLUS} \triangleq \lambda n_1. \lambda n_2. n_1 \text{ SUCC } n_2
\]