Lecture 28
Propositions as Types
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Logics = Type Systems
Inference Rules for Logic

We have used inference rules to build up inductively defined sets of PL concepts: operational steps, valid Hoare triples, associations between terms and types, etc.

Logicians use the same kind of notation to build up the set of true logical formulas.
Inference Rules for Logic

We have used inference rules to build up inductively defined sets of PL concepts: operational steps, valid Hoare triples, associations between terms and types, etc.

Logicians use the same kind of notation to build up the set of true logical formulas.

Here’s a rule from natural deduction, a constructive logic invented by logician Gerhard Gentzen in 1935:

\[
\begin{array}{c}
\phi \\
\hline
\phi \land \psi
\end{array}
\]

^\text{-INTRO}

Given a proof of \( \phi \) and a proof of \( \psi \), the rule lets you construct a proof of \( \phi \land \psi \).
Natural Deduction

Let’s use our usual 4110 tools to define the set of true formulas (“theorems”).
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We’ll start with a grammar for formulas:

\[
\phi ::= \top \\
| \bot \\
| \chi \\
| \phi \land \psi \\
| \phi \lor \psi \\
| \phi \rightarrow \psi \\
| \neg \phi \\
| \forall \chi. \phi
\]

where \(\chi\) ranges over Boolean variables and \(\neg \phi\) is an abbreviation for \(\phi \rightarrow \bot\).
Natural Deduction

Let’s define a judgment that that a formula is true given a set of assumptions $\Gamma$:

$$\Gamma \vdash \phi$$

where $\Gamma$ is just a list of formulas.
Natural Deduction

Let’s define a judgment that a formula is true given a set of assumptions \( \Gamma \):

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\Gamma \vdash \phi
\]

where \( \Gamma \) is just a list of formulas.

If \( \vdash \phi \) (with no assumptions), we say \( \phi \) is a **theorem**.

**Examples:**

- \( \vdash A \wedge B \rightarrow A \)
Natural Deduction

Let’s define a judgment that a formula is true given a set of assumptions $\Gamma$:

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**Examples:**

- $\vdash A \land B \rightarrow A$
- $\vdash \neg(A \land B) \rightarrow \neg A \lor \neg B$
Natural Deduction

Let’s define a judgment that a formula is true given a set of assumptions $\Gamma$:

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If $\vdash \phi$ (with no assumptions), we say $\phi$ is a theorem.

Examples:

- $\vdash A \land B \rightarrow A$
- $\vdash \neg (A \land B) \rightarrow \neg A \lor \neg B$
- $A, B, C \vdash B \lor C$
Natural Deduction

Let’s write the rules for our judgment:

\[
\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi} \quad \land\text{-INTRO}
\]

\[
\frac{\phi \land \psi}{\phi}
\]
Let’s write the rules for our judgment:

\[\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi} \quad \text{\land\text{-INTRO}}\]

\[\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \phi} \quad \text{\land\text{-ELIM1}}\]

\[\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \psi} \quad \text{\land\text{-ELIM2}}\]

\[\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \text{\rightarrow}\]
Let’s write the rules for our judgment:

\[
\begin{align*}
\Gamma \vdash \phi & \quad \Gamma \vdash \psi \\
\hline
\Gamma \vdash \phi \land \psi & \quad \text{\textsc{\textendash intro}}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \phi \land \psi & \\
\hline
\Gamma \vdash \phi & \quad \text{\textsc{\textendash elim1}}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \phi \land \psi & \\
\hline
\Gamma \vdash \psi & \quad \text{\textsc{\textendash elim2}}
\end{align*}
\]

\[
\begin{align*}
\Gamma, \phi \vdash \psi & \\
\hline
\Gamma \vdash \phi \rightarrow \psi & \quad \rightarrow\text{-intro}
\end{align*}
\]
Natural Deduction

Let’s write the rules for our judgment:

\[
\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi} \quad \land\text{-INTRO}
\]

\[
\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \phi} \quad \land\text{-ELIM1}
\]

\[
\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \psi} \quad \land\text{-ELIM2}
\]

\[
\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \rightarrow\text{-INTRO}
\]

...and so on.
Natural Deduction

**Axiom**

\[ \Gamma, \phi \vdash \phi \]

**→-INTRO**

\[ \Gamma, \phi \vdash \psi \quad \Gamma \vdash \phi \rightarrow \psi \]

**→-ELIM**

\[ \Gamma \vdash \phi \rightarrow \psi \quad \Gamma \vdash \phi \]

**∧-INTRO**

\[ \Gamma \vdash \phi \quad \Gamma \vdash \psi \]

\[ \Gamma \vdash \phi \land \psi \]

**∧-ELIM1**

\[ \Gamma \vdash \phi \land \psi \]

\[ \Gamma \vdash \phi \]

\[ \Gamma \vdash \psi \]

**∧-ELIM2**

\[ \Gamma \vdash \phi \land \psi \]

\[ \Gamma \vdash \psi \]

**∨-INTRO1**

\[ \Gamma \vdash \phi \]

\[ \Gamma \vdash \phi \lor \psi \]

**∨-INTRO2**

\[ \Gamma \vdash \psi \]

\[ \Gamma \vdash \phi \lor \psi \]

**∨-ELIM**

\[ \Gamma \vdash \phi \lor \psi \quad \Gamma \vdash \phi \rightarrow \chi \]

\[ \Gamma \vdash \psi \rightarrow \chi \]

\[ \Gamma \vdash \chi \]

**∀-INTRO**

\[ \Gamma, \rho \vdash \phi \]

\[ \Gamma \vdash \forall P. \phi \]

**∀-ELIM**

\[ \Gamma \vdash \forall P. \phi \]

\[ \Gamma \vdash \phi\{\psi/P\} \]
Natural Deduction

Let’s try a proof! We can write a proof that $A \land B \rightarrow B \land A$ is a theorem.

\[\begin{align*}
A \land B & \rightarrow A \land B \\
\hline
A \land B + A \land B & \rightarrow A \land B + A \land B \\
\hline
A \land B + B & \rightarrow A \land B + A \\
\hline
A \land B + A & \rightarrow A \land B + A \\
\hline
A \land B + B & \rightarrow A \land B + B \\
\hline
A \land B & \rightarrow B \land A \\
\hline
\end{align*}\]
Natural Deduction

Let’s try a proof! We can write a proof that $A \land B \rightarrow B \land A$ is a theorem.

\[
\begin{align*}
&\quad A \land B \vdash A \land B & \text{AXIOM} \\
&\quad A \land B \vdash B & \text{\land-ELIM2} \\
&\quad A \land B \vdash A & \text{\land-ELIM1} \\
&\quad A \land B \vdash B \land A & \text{\land-INTRO}
\end{align*}
\]

\[
\begin{align*}
\quad &\quad A \land B \vdash B \land A & \text{\land-INTRO} \\
\quad &\quad \vdash A \land B \rightarrow B \land A & \text{\rightarrow-INTRO}
\end{align*}
\]
Natural Deduction

Let’s try a proof! We can write a proof that $A \land B \rightarrow B \land A$ is a theorem.

\[
\begin{align*}
A \land B & \vdash A \land B & \text{AXIOM} \\
A \land B & \vdash A & \text{\land-ELIM2} \\
A \land B & \vdash B & \text{\land-ELIM1} \\
A \land B & \vdash B \land A & \text{\land-INTRO} \\
\vdash A \land B & \rightarrow B \land A & \rightarrow\text{-INTRO}
\end{align*}
\]

Does this look familiar?
Natural Deduction

Let’s try a proof! We can write a proof that $A \land B \rightarrow B \land A$ is a theorem.

$$\begin{align*}
A \land B & \vdash A \land B \\
\text{AXIOM} & \\
A \land B & \vdash A \land B \\
\wedge\text{-ELIM2} & \\
A \land B & \vdash A \\
\text{AXIOM} & \\
A \land B & \vdash B \\
\wedge\text{-ELIM2} & \\
A \land B & \vdash B \land A \\
\text{-INTRO} & \\
\vdash A \land B \rightarrow B \land A & \\
\text{→-INTRO} & \\
\end{align*}$$

Does this look familiar?

$$\begin{align*}
x: A \times B & \vdash x: A \times B \\
\text{T-VAR} & \\
x: A \times B & \vdash x: A \times B \\
\text{T-#1} & \\
x: A \times B & \vdash x: B \\
\text{T-#2} & \\
x: A \times B & \vdash x: A \\
\text{T-PAIR} & \\
x: A \times B & \vdash x: B \times A \\
\text{T-ABS} & \\
\vdash (\#1 \rightarrow \#2) & \\
\end{align*}$$
Propositions as Types

Every natural deduction proof tree has a corresponding type tree in System F with product and sum types! And vice-versa!

<table>
<thead>
<tr>
<th>Type Systems</th>
<th>Formal Logic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$ Type</td>
<td>$\phi$ Formula</td>
</tr>
<tr>
<td>$\tau$ is inhabited</td>
<td>$\phi$ is a theorem</td>
</tr>
<tr>
<td>e Well-typed exp.</td>
<td>$\pi$ Proof</td>
</tr>
</tbody>
</table>

A program with a given type acts as a *witness* that the type’s corresponding formula is true.
Propositions as Types

Every type rule in System F with product and sum types corresponds 1-1 with a proof rule in natural deduction:

<table>
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<tr>
<td>→</td>
<td>→ Implication</td>
</tr>
<tr>
<td>×</td>
<td>∧ Conjunction</td>
</tr>
<tr>
<td>+</td>
<td>∨ Disjunction</td>
</tr>
<tr>
<td>∀ Universal</td>
<td>∀ Quantifier</td>
</tr>
</tbody>
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You can even add existential types to correspond to existential quantification. It still works!
Propositions as Types

Every type rule in System F with product and sum types corresponds 1-1 with a proof rule in natural deduction:

<table>
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<tbody>
<tr>
<td>→ Function</td>
<td>→ Implication</td>
</tr>
<tr>
<td>× Product</td>
<td>∧ Conjunction</td>
</tr>
<tr>
<td>+ Sum</td>
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<tr>
<td>∀ Universal</td>
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You can even add existential types to correspond to existential quantification. It still works!

Is this a coincidence? Natural deduction was invented by a German logician in 1935. Types for the $\lambda$-calculus were invented by Church at Princeton in 1940.
<table>
<thead>
<tr>
<th>Propositions as Types Through the Ages</th>
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<tbody>
<tr>
<td><strong>Natural Deduction</strong></td>
</tr>
<tr>
<td>Gentzen (1935)</td>
</tr>
<tr>
<td>↔  <strong>Typed λ-Calculus</strong></td>
</tr>
<tr>
<td>Church (1940)</td>
</tr>
<tr>
<td><strong>Type Schemes</strong></td>
</tr>
<tr>
<td>Hindley (1969)</td>
</tr>
<tr>
<td>↔  <strong>ML’s Type System</strong></td>
</tr>
<tr>
<td>Milner (1975)</td>
</tr>
<tr>
<td><strong>System F</strong></td>
</tr>
<tr>
<td>Girard (1972)</td>
</tr>
<tr>
<td>↔  <strong>Polymorphic λ-Calculus</strong></td>
</tr>
<tr>
<td>Reynolds (1974)</td>
</tr>
<tr>
<td><strong>Modal Logic</strong></td>
</tr>
<tr>
<td>Lewis (1910)</td>
</tr>
<tr>
<td>↔  <strong>Monads</strong></td>
</tr>
<tr>
<td>Kleisli (1965), Moggi (1987)</td>
</tr>
<tr>
<td><strong>Classical–Intuitionistic Embedding</strong></td>
</tr>
<tr>
<td>Gödel (1933)</td>
</tr>
<tr>
<td>↔  <strong>Continuation Passing Style</strong></td>
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Term Assignment

This all means that we have a new way of proving theorems: writing programs!
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To prove a formula $\phi$:

1. Convert the $\phi$ into its corresponding type $\tau$.
2. Find some program $\nu$ that has the type $\tau$.
3. Realize that the existence of $\nu$ implies a type tree for $\vdash \nu : \tau$, which implies a proof tree for $\vdash \phi$.

$$\forall \chi. \chi \equiv 1$$
Negation and Continuations

Let’s explore one extension. We’d like to use this rule from classical logic:

\[ \frac{\Gamma \vdash \phi}{\Gamma \vdash \neg \neg \phi} \]

but there’s no obvious correspondence in System F.
Negation and Continuations

Let’s explore one extension. We’d like to use this rule from classical logic:

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \neg \neg \phi}$$

but there’s no obvious correspondence in System F.

Recall that $\neg \phi$ is shorthand for $\phi \rightarrow \bot$. So $\neg \neg \phi$ corresponds to the System F function type $(\tau \rightarrow \bot) \rightarrow \bot$.

So what we need is a way to take any program of any type $\tau$ and turn it into a program of type $(\tau \rightarrow \bot) \rightarrow \bot$. 
Negation and Continuations

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\begin{align*}
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\Gamma & \vdash \neg \neg \phi
\end{align*}
\]

but there’s no obvious correspondence in System F.

Recall that \( \neg \phi \) is shorthand for \( \phi \rightarrow \bot \). So \( \neg \neg \phi \) corresponds to the System F function type \((\tau \rightarrow \bot) \rightarrow \bot\).

So what we need is a way to take any program of any type \( \tau \) and turn it into a program of type \((\tau \rightarrow \bot) \rightarrow \bot\).

Shockingly, that’s exactly what the CPS transform does! A \( \tau \) becomes a function that takes a continuation \( \tau \rightarrow \bot \) and invokes it, producing \( \bot \).