Lecture 2
Introduction to Semantics
Question: What is the meaning of a program?
Semantics

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Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...

...but none of these is a satisfactory solution.
Formal Semantics

Three Approaches

- **Operational**
  - Model program by execution on abstract machine
  - Useful for implementing compilers and interpreters

- **Denotational:**
  - Model program as mathematical objects
  - Useful for theoretical foundations

- **Axiomatic**
  - Model program by the logical formulas it obeys
  - Useful for proving program correctness

\[ \langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \]

\[ [e] \]

\[ \vdash \{ \phi \} e \{ \psi \} \]
Arithmetic Expressions
Syntax

A language of integer arithmetic expressions with assignment.
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A language of integer arithmetic expressions with assignment.

Metavariables:

\[ x, y, z \in \text{Var} = \{a, b, c, \ldots\} \]
\[ n, m \in \text{Int} = \mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\} \]
\[ e \in \text{Exp} \]

\[ e, e_2 \]

\[ \text{Var} \cap \text{Int} = \emptyset \]

\[ x + y \]

\[ x + x \]
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\[ x, y, z \in \text{Var} \]
\[ n, m \in \text{Int} \]
\[ e \in \text{Exp} \]

**Backus-Naur Form**

BNF Grammar:

\[ e ::= x \]
\[ n \]
\[ e_1 + e_2 \]
\[ e_1 \times e_2 \]
\[ x := e_1 ; e_2 \]

= \{ \text{a, b, c, ...}, \text{1, 2, 3, ...}, \text{-1, -2, -3, ...}, \text{a + 5, 6 + 2, 12 + 6 + a}, \ldots \} \]
What expression does the string “1 + (2 * 3)” describe?
Ambiguity

What expression does the string “1 + 2 * 3” describe?
There are two possible parse trees:

```
+  *
\  \  /  \  /  \  /
1  2  *  3  +  1  2
   \  \  /  \  /  \  /
    2  3  3  1  2
```
Ambiguity

What expression does the string “1 + 2 * 3” describe?
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In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).
Representing Expressions

BNF Grammar:

\[ e ::= x \]

\[ \quad | \quad n \]

\[ \quad | \quad e_1 + e_2 \]

\[ \quad | \quad e_1 * e_2 \]

\[ \quad | \quad x := e_1 ; e_2 \]
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OCaml:

```
type exp = Var of string
| Int of int
| Add of exp * exp
| Mul of exp * exp
| Assgn of string * exp * exp
```

Example: `Mul(Int 2, Add(Var "foo", Int 1))`
Representing Expressions

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Java:

```java
abstract class Expr { }
class Var extends Expr { String name; ... }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assgn extends Expr { String var, Expr exp1, exp2; ... }
```

Example: new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))
Quiz

• $7 + (4 \times 2)$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := (6 + 1); (2 \times (3 \times i))$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1 ; \ 2 \times 3 \times i$ evaluates to 42
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1; 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1; \ 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to error?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1 \; \text{;} \; 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to error?

The rest of this lecture will make these intuitions precise...
Mathematical Preliminaries
The *product* of two sets $A$ and $B$, written $A \times B$, contains all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$.

\[
A \times B = \left\{ (a, b) \mid a \in A, \ b \in B \right\}
\]
Binary Relations

The *product* of two sets $A$ and $B$, written $A \times B$, contains all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$.

A *binary relation* on $A$ and $B$ is just a subset $R \subseteq A \times B$. 
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**Some Important Relations**

- empty – $\emptyset$
- total – $A \times B$
- identity on $A$ – $\{(a, a) \mid a \in A\}$.
- composition $R; S$ – $\{(a, c) \mid \exists b. (a, b) \in R \land (b, c) \in S\}$
A *(total)* function \( f \) is a binary relation \( f \subseteq A \times B \) with the property that every \( a \in A \) is related to exactly one \( b \in B \).
Functions

A (total) function $f$ is a binary relation $f \subseteq A \times B$ with the property that every $a \in A$ is related to exactly one $b \in B$. When $f$ is a function, we usually write $f : A \to B$ instead of $f \subseteq A \times B$. 
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The domain and range of \( f \) are defined the same way as for relations
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The image of \( f \) is the set of elements \( b \in B \) that are mapped to by at least one \( a \in A \). More formally: image\((f) \triangleq \{f(a) \mid a \in A\} \)
Some Important Functions

Given two functions $f : A \rightarrow B$ and $g : B \rightarrow C$, the composition of $f$ and $g$ is defined by: $(g \circ f)(x) = g(f(x))$  

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A partial function $f : A \rightharpoonup B$ is a total function $f : A' \to B$ on a set $A' \subseteq A$. The notation $\text{dom}(f)$ refers to $A'$. 
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A function $f : A \to B$ is said to be \textit{injective} (or \textit{one-to-one}) if and only if $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. 
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A function \( f : A \rightarrow B \) is said to be surjective (or onto) if and only if the image of \( f \) is \( B \).
Operational Semantics
Overview

An **operational semantics** describes how a program executes on some (typically idealized) abstract machine.
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For our language, a **configuration** \( \langle \sigma, e \rangle \) has two components:

- a **store** \( \sigma \) that records the values of variables
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- and the **expression** \( e \) being evaluated

More formally,

\[
\text{Store} \triangleq \text{Var} \rightarrow \text{Int} \\
\text{Config} \triangleq \text{Store} \times \text{Exp}
\]

Note that a store is a **partial** function from variables to integers.
Operational Semantics

The small-step operational semantics itself is a relation on configurations—i.e., a subset of \( \text{Config} \times \text{Config} \).
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Notation: $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$
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**Question:** How should we define this relation?
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**Answer:** define it inductively, using **inference rules**:

$$p = m + n$$

$$\frac{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle}{\text{ADD}}$$
Operational Semantics

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**Answer:** define it inductively, using **inference rules**:

\[
\begin{align*}
\text{ADD} & \\
\frac{p = m + n}{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle}
\end{align*}
\]

Intuitively, if facts above the line hold, then facts below the line hold. More formally, “$\rightarrow$” is the smallest relation “closed” under the inference rules.
Variables

\[
\begin{align*}
n &= \sigma(x) \\
\langle \sigma, x \rangle &\rightarrow \langle \sigma, n \rangle^
\end{align*}
\]
Addition

\[
\begin{align*}
\langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e'_1 \rangle \\
\langle \sigma, e_1 + e_2 \rangle & \rightarrow \langle \sigma', e'_1 + e_2 \rangle
\end{align*}
\]

\text{LADD}
Addition

\[ \langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle \]  \quad \text{LADD} \\
\[ \langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e'_1 + e_2 \rangle \] \\
\[ \langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle \]  \quad \text{RADD} \\
\[ \langle \sigma, n + e_2 \rangle \rightarrow \langle \sigma', n + e'_2 \rangle \]
Addition

\[
\begin{align*}
\langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e'_1 \rangle & \text{LADD} \\
\langle \sigma, e_1 + e_2 \rangle & \rightarrow \langle \sigma', e'_1 + e_2 \rangle \\
\langle \sigma, e_2 \rangle & \rightarrow \langle \sigma', e'_2 \rangle & \text{RADD} \\
\langle \sigma, n + e_2 \rangle & \rightarrow \langle \sigma', n + e'_2 \rangle \\
\langle \sigma, n + m \rangle & \rightarrow \langle \sigma, p \rangle & \text{ADD}
\end{align*}
\]
Multiplication

\[
\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle
\]

\[
\langle \sigma, e_1 \ast e_2 \rangle \rightarrow \langle \sigma', e'_1 \ast e_2 \rangle
\]

LMUL
Multiplication

\[
\begin{array}{c}
\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle \\
\langle \sigma, e_1 \ast e_2 \rangle \rightarrow \langle \sigma', e'_1 \ast e_2 \rangle \\
\end{array}
\]

LMUL

\[
\begin{array}{c}
\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle \\
\langle \sigma, n \ast e_2 \rangle \rightarrow \langle \sigma', n \ast e'_2 \rangle \\
\end{array}
\]

RMUL
Multiplication

\[
\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, e_1 \times e_2 \rangle \rightarrow \langle \sigma', e'_1 \times e_2 \rangle} \quad \text{LMUL}
\]

\[
\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle}{\langle \sigma, n \times e_2 \rangle \rightarrow \langle \sigma', n \times e'_2 \rangle} \quad \text{RMUL}
\]

\[
p = m \times n \quad \frac{\langle \sigma, m \times n \rangle \rightarrow \langle \sigma, p \rangle}{\text{MUL}}
\]
Assignment

\[
\begin{align*}
\langle \sigma, e_1 \rangle &\rightarrow \langle \sigma', e'_1 \rangle \\
\langle \sigma, x := e_1 ; e_2 \rangle &\rightarrow \langle \sigma', x := e'_1 ; e_2 \rangle 
\end{align*}
\]

AssGN1
Assignment

\[
\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle \quad \text{ASSGN1}
\]

\[
\langle \sigma, x := e_1 ; e_2 \rangle \rightarrow \langle \sigma', x := e'_1 ; e_2 \rangle
\]

\[
\sigma' = \sigma[x \mapsto n] \quad \text{ASSGN}
\]

\[
\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle
\]

Notation: \( \sigma[x \mapsto n] \) maps \( x \) to \( n \) and otherwise behaves like \( \sigma \)
Operational Semantics

\[
\frac{n = \sigma(x)}{\langle \sigma, x \rangle \to \langle \sigma, n \rangle} \quad \text{VAR}
\]

\[
\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \to \langle \sigma', e_1' + e_2 \rangle} \quad \text{LADD}
\]

\[
\frac{\langle \sigma, e_2 \rangle \to \langle \sigma', e_2' \rangle}{\langle \sigma, n + e_2 \rangle \to \langle \sigma', n + e_2' \rangle} \quad \text{RADD}
\]

\[
\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 \star e_2 \rangle \to \langle \sigma', e_1' \star e_2 \rangle} \quad \text{LMUL}
\]

\[
\frac{\langle \sigma, e_2 \rangle \to \langle \sigma', e_2' \rangle}{\langle \sigma, n \star e_2 \rangle \to \langle \sigma', n \star e_2' \rangle} \quad \text{RMUL}
\]

\[
\frac{p = m + n}{\langle \sigma, n + m \rangle \to \langle \sigma, p \rangle} \quad \text{ADD}
\]

\[
\frac{p = m \times n}{\langle \sigma, m \star n \rangle \to \langle \sigma, p \rangle} \quad \text{MUL}
\]

\[
\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, x := e_1 \ ; \ e_2 \rangle \to \langle \sigma', x := e_1' \ ; \ e_2 \rangle} \quad \text{ASSIGN1}
\]