Propositions as Types

Logics = Type Systems
Constructive Logic

Let’s start with constructive logic, where the rules work like functions that take smaller proofs and generate larger proofs.
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Here’s a rule from natural deduction, a constructive logic invented by logician Gerhard Gentzen in 1935:

\[
\frac{\phi \quad \psi}{\phi \land \psi} \quad \land\text{-INTRO}
\]

Given a proof of \( \phi \) and a proof of \( \psi \), it lets you construct a proof of \( \phi \land \psi \).
Natural Deduction

In natural deduction, we define the set of true formulas ("theorems") inductively.
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We’ll start with a grammar for formulas:

\[ \phi ::= \top \]

\[ \bot \]

\[ X \]

\[ \phi \land \psi \]

\[ \phi \lor \psi \]

\[ \phi \rightarrow \psi \]

\[ \neg \phi \]

\[ \forall x. \phi \]

where \( X \) ranges over Boolean variables and \( \neg \phi \) is an abbreviation for \( \phi \rightarrow \bot \).
Natural Deduction

Let’s define a judgment that a formula is true given a set of assumptions $\Gamma$:

$$\Gamma \vdash \phi$$

where $\Gamma$ is just a list of formulas.
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If $\vdash \phi$ (with no assumptions), we say $\phi$ is a theorem.

Examples:
- $\vdash A \land B \rightarrow A$
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**Examples:**

- $\vdash A \land B \rightarrow A$
- $\vdash \neg (A \land B) \rightarrow \neg A \lor \neg B$
Natural Deduction

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Examples:

- $\vdash A \land B \rightarrow A$
- $\vdash \neg(A \land B) \rightarrow \neg A \lor \neg B$
- $A, B, C \vdash B$
Natural Deduction

Let’s write the rules for our judgment:

\[
\Gamma \vdash \phi \quad \Gamma \vdash \psi \\
\hline
\Gamma \vdash \phi \land \psi
\]

\(\land\text{-INTRO}\)
Natural Deduction

Let’s write the rules for our judgment:

\[
\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi} \quad \land\text{-INTRO}
\]

\[
\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \phi} \quad \land\text{-ELIM1}
\]

\[
\frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \psi} \quad \land\text{-ELIM2}
\]
Natural Deduction

Let’s write the rules for our judgment:

\[\Gamma \vdash \phi \quad \Gamma \vdash \psi \quad \Gamma \vdash \phi \land \psi \quad \text{\Leftrightarrow} \text{INTRO} \]

\[\Gamma \vdash \phi \land \psi \quad \Gamma \vdash \phi \quad \text{\Leftrightarrow} \text{ELIM1} \]

\[\Gamma \vdash \phi \land \psi \quad \Gamma \vdash \psi \quad \text{\Leftrightarrow} \text{ELIM2} \]

\[\Gamma, \phi \vdash \psi \quad \Gamma \vdash \phi \rightarrow \psi \quad \text{\Leftrightarrow} \text{INTRO} \]
Natural Deduction

Let’s write the rules for our judgment:

\[ \frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi} \quad \land\text{-INTRO} \]

\[ \frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \phi} \quad \land\text{-ELIM1} \]

\[ \frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \psi} \quad \land\text{-ELIM2} \]

\[ \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \quad \rightarrow\text{-INTRO} \]

...and so on.
Natural Deduction

\[ \Gamma, \phi \vdash \phi \text{ Axiom} \]

\[ \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \rightarrow\text{-INTRO} \]

\[ \frac{\Gamma \vdash \phi \rightarrow \psi}{\Gamma \vdash \phi} \rightarrow\text{-ELIM} \]

\[ \frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \phi} \land\text{-INTRO} \]

\[ \frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \psi} \land\text{-ELIM1} \]

\[ \frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \phi} \land\text{-ELIM2} \]

\[ \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \lor \psi} \lor\text{-INTRO1} \]

\[ \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \lor \psi} \lor\text{-INTRO2} \]

\[ \frac{\Gamma \vdash \phi \lor \psi}{\Gamma \vdash \phi \rightarrow \chi} \lor\text{-ELIM1} \]

\[ \frac{\Gamma \vdash \psi \rightarrow \chi}{\Gamma \vdash \phi \rightarrow \chi} \lor\text{-ELIM2} \]

\[ \frac{\Gamma, P \vdash \phi}{\Gamma \vdash \forall P. \phi} \forall\text{-INTRO} \]

\[ \frac{\Gamma \vdash \forall P. \phi}{\Gamma \vdash \phi \{\psi/P\}} \forall\text{-ELIM} \]
Let’s try a proof! Here’s a proof that $A \land B \rightarrow B \land A$ is a theorem.

\[
\begin{align*}
& A \land B \vdash A \land B \quad \text{AXIOM} \\
& A \land B \vdash A \quad \text{\land-ELIM2} \\
& A \land B \vdash B \quad \text{\land-ELIM1} \\
& A \land B \vdash B \land A \quad \text{\land-INTRO} \\
& A \land B \vdash B \land A \\
& \vdash A \land B \rightarrow B \land A \quad \rightarrow\text{-INTRO}
\end{align*}
\]
Natural Deduction

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\begin{align*}
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\vdash & A \land B & \text{-ELIM2} \\
A \land B \vdash & B & \text{AXIOM} \\
\vdash & A \land B \vdash B \land A & \text{-INTRO} \\
\vdash & A \land B \vdash B \land A & \text{-INTRO} \\
\vdash & A \land B \rightarrow B \land A & \rightarrow\text{-INTRO}
\end{align*}
\]

Doesn’t this look a little... familiar?

\[
\begin{align*}
x: A \times B \vdash & x: A \times B & \text{T-VAR} \\
\vdash & x: A \times B \vdash \#2 x: B & \text{T-#1} \\
\vdash & x: A \times B \vdash \#1 x: A & \text{T-#2} \\
\vdash & x: A \times B \vdash (\#2 x, \#1 x): B \times A & \text{T-PAIR} \\
\vdash & \lambda x. (\#2 x, \#1 x): A \times B \rightarrow B \times A & \text{T-ABS}
\end{align*}
\]
Propositions as Types

Every natural deduction proof tree has a corresponding type tree in System F with product and sum types! And vice-versa!

<table>
<thead>
<tr>
<th>Type Systems</th>
<th>Formal Logic</th>
</tr>
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<tbody>
<tr>
<td>( \tau ) Type</td>
<td>( \phi ) Formula</td>
</tr>
<tr>
<td>( \tau ) is inhabited</td>
<td>( \phi ) is a theorem</td>
</tr>
<tr>
<td>( e ) Well-typed expression</td>
<td>( \pi ) Proof</td>
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A program with a given type acts as a *witness* that the type’s corresponding formula is true.
Propositions as Types

Every type rule in System F with product and sum types corresponds 1-1 with a proof rule in natural deduction:

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<td>∧ Conjunction</td>
</tr>
<tr>
<td>+</td>
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You can even add existential types to correspond to existential quantification. It still works!
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Is this a coincidence? Natural deduction was invented by a German logician in 1935. Types for the λ-calculus were invented by Church at Princeton in 1940.
<table>
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<td><strong>Classical–Intuitionistic Embedding</strong></td>
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<td><strong>Typed λ-Calculus</strong></td>
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Term Assignment

This all means that we have a new way of proving theorems: writing programs!
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To prove a formula $\phi$:

1. Convert the $\phi$ into its corresponding type $\tau$.
2. Find some program $\nu$ that has the type $\tau$.
3. Realize that the existence of $\nu$ implies a type tree for $\vdash \nu : \tau$, which implies a proof tree for $\vdash \phi$. 
Let’s explore one extension. We’d like to use this rule from classical logic:

\[
\begin{array}{c}
\Gamma \vdash \phi \\
\hline
\Gamma \vdash \neg \neg \phi
\end{array}
\]

but there’s no obvious correspondence in System F.
Negation and Continuations

Let’s explore one extension. We’d like to use this rule from classical logic:

\[
\frac{\Gamma \vdash \phi}{\Gamma \vdash \neg \neg \phi}
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but there’s no obvious correspondence in System F.

Recall that \(\neg \phi\) is shorthand for \(\phi \rightarrow \bot\). So \(\neg \neg \phi\) corresponds to the System F function type \((\tau \rightarrow \bot) \rightarrow \bot\).

So what we need is a way to take any program of any type \(\tau\) and turn it into a program of type \((\tau \rightarrow \bot) \rightarrow \bot\).
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So what we need is a way to take any program of any type \( \tau \) and turn it into a program of type \( (\tau \rightarrow \bot) \rightarrow \bot \).

Shockingly, that’s exactly what the CPS transform does! A \( \tau \) becomes a function that takes a continuation \( \tau \rightarrow \bot \) and invokes it, producing \( \bot \).