Announcements

- New TA with new office hours (welcome back, Andrew!)
  - Monday usually; Friday this week
- Homework 2 returned
  - Out of 36, $\bar{x} = 28.9$, $\sigma = 6.2$, median 30
A Recipe for Induction Over Derivations

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   d. Does the goal have premises from the same relation? If not, this is a base case. Reason directly.
   e. If so, this is an inductive case. Apply $P$ to those subderivations you marked with vertical dots. Write down the resulting conclusion. Use that fact to prove $P(D)$ for this derivation.
Overview

Last time
- Hoare Logic

Today
- “Decorated” programs
- Weakest Preconditions
Review: Hoare Logic

\[
\begin{align*}
\vdash \{P\} \text{skip} \{P\} & \quad \text{SKIP} \quad \vdash \{P[a/x]\} x := a \{P\} \quad \text{ASSIGN} \\
\vdash \{P\} c_1 \{R\} & \quad \vdash \{R\} c_2 \{Q\} \quad \vdash \{P\} c_1; c_2 \{Q\} \quad \text{SEQ} \\
\vdash \{P \land b\} c_1 \{Q\} & \quad \vdash \{P \land \neg b\} c_2 \{Q\} \quad \vdash \{P\} \text{if } b \text{ then } c_1 \text{ else } c_2 \{Q\} \quad \text{IF} \\
\vdash \{P \land b\} c \{P\} & \quad \vdash \{P\} \text{while } b \text{ do } c \{P \land \neg b\} \quad \text{WHILE} \\
\models P \Rightarrow P' & \quad \vdash \{P'\} c \{Q'\} \quad \models Q' \Rightarrow Q \quad \text{CONSEQUENCE}
\end{align*}
\]
Decorated Programs

**Observation:** Once we’ve identified loop invariants and uses of consequence, the structure of a Hoare logic is determined!

**Notation:** Can write proofs by “decorating” programs with:

- A precondition ($\{P\}$)
- A postcondition ($\{Q\}$)
- Invariants ($\{I\}$ \textbf{while} $b$ \textbf{do} $c$)
- Uses of consequence $\{R\} \Rightarrow \{S\}$
- Assertions between sequences $c_1; \{T\}c_2$

A decorated program describes a valid Hoare logic proof if the rest of the proof tree’s structure is implied. (Caveats: Invariants are constrained, etc.)
Example: Decorated Factorial

\{x = n \land n > 0\}

\begin{align*}
y &:= 1; \\
\textbf{while } x > 0 \textbf{ do } \{ \\
\quad &y := y \times x; \\
\quad &x := x - 1 \\
\} \\
\{y = n!\} 
\end{align*}
Example: Decorated Factorial

\{x = n \land n > 0\} \Rightarrow
\{1 = 1 \land x = n \land n > 0\}
y := 1;
\{y = 1 \land x = n \land n > 0\} \Rightarrow
\{y \times x! = n! \land x \geq 0\}

while \(x > 0\) do {
\{y \times x! = n! \land x > 0 \land x \geq 0\} \Rightarrow
\{y \times x \times (x - 1)! = n! \land (x - 1) \geq 0\}
y := y \times x;
\{y \times (x - 1)! = n! \land (x - 1) \geq 0\}
x := x - 1
\{y \times x! = n! \land x \geq 0\}
\}
\{y \times x! = n! \land (x \geq 0) \land \neg (x > 0)\} \Rightarrow
\{y = n!\}
Informal Rules for Decoration

Check whether a decorated program represents a valid proof using local consistency checks.
Informal Rules for Decoration

Check whether a decorated program represents a valid proof using local consistency checks.

For `skip`, the precondition and postcondition should be the same:

\[
\begin{align*}
\{P\} \\
\text{skip} \\
\{P\}
\end{align*}
\]
Informal Rules for Decoration

For sequences, \( \{P\} c_1 \{R\} \) and \( \{R\} c_2 \{Q\} \) must be (recursively) locally consistent:

\[
\begin{align*}
\{P\} \\
c_1; \\
\{R\} \\
c_2 \\
\{Q\}
\end{align*}
\]
Informal Rules for Decoration

Assignment should use the substitution from the rule:

\[
\{P[a/x]\}
\]

\[
x := a
\]

\[
\{P\}
\]
Informal Rules for Decoration

An if is locally consistent when both branches are locally consistent after adding the branch condition to each:

\[
\begin{align*}
\{P\} \\
\textbf{if } b \textbf{ then} & \quad \{P \land b\} \\
& \quad c_1 \\
\quad \{Q\} \\
\textbf{else} & \quad \{P \land \neg b\} \\
& \quad c_2 \\
& \quad \{Q\} \\
& \quad \{Q\}
\end{align*}
\]
Informal Rules for Decoration

Decorate a **while** with the loop invariant:

\[
\{ P \} \\
\textbf{while } b \textbf{ do} \\
\{ P \land b \} \\
c \\
\{ P \} \\
\{ P \land \neg b \}\]
Informal Rules for Decoration

To capture the CONSEQUENCE rule, you can always write a (valid) implication:

\[ \{P\} \implies \{Q\} \]
Example

\[
\begin{align*}
\{ & \quad \{ \} \\
\textbf{while} (0 < y) \ & \textbf{do} ( \\
& \quad x := x + 1; \\
& \quad y := y - 1 \\
& ) \} \ \ \\
\{ & \quad \} 
\end{align*}
\]
Example

\{x = m \land y = n \land 0 \leq n\}

**while** \((0 < y)\) **do** (  
    \[x := x + 1;\]
    \[y := y - 1\]
  )  

\{x = m + n\}
Example

\{x = m \land y = n \land 0 \leq n\} \Rightarrow \{/\}

\textbf{while} (0 < y) \textbf{do} (  
  \{l \land 0 < y\} \Rightarrow
  \{l[y - 1/y][x + 1/x]\}
  x := x + 1;
  \{l[y - 1/y]\}
  y := y - 1
  \{/\}
)
\{l \land 0 \not< y\} \Rightarrow
\{x = m + n\}

Where \(l\) is \((x = m + n - y) \land 0 \leq y\).
while (x ≠ 0) do (x := x - 1)
Example

\{true\}

\textbf{while} (x \neq 0) \textbf{do} ( \\
\hspace{2em} x := x - 1 \\
) \\
\{x = 0\}
Example

{ }

y := 1

while (0 < x) do (  
    x := x - 1;
    y := y * 2
)

{ }
Example

\{ x = n \land 0 \leq n \} \\

y := 1 \\
**while** (0 < x) **do** ( \\
    x := x - 1; \\
    y := y * 2 \\
) \\

\{ y = 2^n \}