Lecture 2
Introduction to Semantics

26 August 2016
29 August 2012
Announcements

**Wednesday Lecture**
- Moved to Thurston 203

**Kozen Foster Office Hours**
- Today 11a–12pm in Gates 432
- 10–11am Gates 436

**Mota Office Hours**
- Wed 11am–12pm in TBD
- Thurs 2:30pm–4pm in TBD

**Homework #1**
- Out: Wednesday, September 3rd
- Due: Wednesday, September 10th
- Distributed via CMS
Semantics

**Question:** What is the meaning of a program?
Semantics

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Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...

...but none of these is a satisfactory solution.
Formal Semantics

Three Approaches

- **Operational**
  - Model program by execution on abstract machine
  - Useful for implementing compilers and interpreters
  \[ \langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \]

- **Denotational**: \[[e]\]
  - Model program as mathematical objects
  - Useful for theoretical foundations

- **Axiomatic**: \( \vdash \{ \phi \} e \{ \psi \} \)
  - Model program by the logical formulas it obeys
  - Useful for proving program correctness
Arithmetic Expressions
Syntax

A language of integer arithmetic expressions with assignment.
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Metavariables:

\[ x, y, z \in \text{Var} \]
\[ n, m \in \text{Int} \]
\[ e \in \text{Exp} \]
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\[ n, m \in \text{Int} \]
\[ e \in \text{Exp} \]

BNF Grammar:

\[ e ::= x \]
\[ | n \]
\[ | e_1 + e_2 \]
\[ | e_1 \times e_2 \]
\[ | x := e_1 ; e_2 \]
Ambiguity

What expression does the string “1 + 2 * 3” describe?
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There are two possible parse trees:
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In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).
Representing Expressions

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  | n \\
  | e_1 + e_2 \\
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  | x := e_1 ; e_2
\]
Representing Expressions

BNF Grammar:

\[ e ::= x \]
\[ | n \]
\[ | e_1 + e_2 \]
\[ | e_1 * e_2 \]
\[ | x := e_1 ; e_2 \]

OCaml:

```ocaml
type exp =
  | Var of string
  | Int of int
  | Add of exp * exp
  | Mul of exp * exp
  | Assgn of string * exp * exp
```

Example: Mul(Int 2, Add(Var "foo", Int 1))
Representing Expressions

BNF Grammar:

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\[ | e_1 \ast e_2 \]
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Java:

```java
abstract class Expr { }
class Var extends Expr { String name; ... }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assgn extends Expr { String var, Expr exp1, exp2; ... }
```

Example: new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))
Quiz

• $7 + (4 \times 2)$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to $15$
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1 \ ; \ 2 \times 3 \times i$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1; \ 2 \times 3 \times i$ evaluates to 42
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1 ; 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1 \; ; \; 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to error?
Quiz

• $7 + (4 \times 2)$ evaluates to 15
• $i := 6 + 1 \; ; \; 2 \times 3 \times i$ evaluates to 42
• $x + 1$ evaluates to error?

The rest of this lecture will make these intuitions precise...
Mathematical Preliminaries
The *product* of two sets $A$ and $B$, written $A \times B$, contains all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$. 
Binary Relations

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**Some Important Relations**

- empty – $\emptyset$
- total – $A \times B$
- identity on $A$ – $\{(a, a) \mid a \in A\}$.
- composition $R; S$ – $\{(a, c) \mid \exists b. (a, b) \in R \land (b, c) \in S\}$
A (total) function $f$ is a binary relation $f \subseteq A \times B$ with the property that every $a \in A$ is related to exactly one $b \in B$. More formally, the image of $f$ is defined as $\{b \in B \mid \exists a \in A : (a, b) \in f\}$. This means that for every element in the domain $A$, there is exactly one corresponding element in the range $B$.
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When $f$ is a function, we usually write $f : A \to B$ instead of $f \subseteq A \times B$
Functions

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The domain and range of $f$ are defined the same way as for relations.
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The domain and range of $f$ are defined the same way as for relations

The image of $f$ is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. More formally: $\text{image}(f) \triangleq \{ f(a) \mid a \in A \}$
Given two functions $f : A \rightarrow B$ and $g : B \rightarrow C$, the composition of $f$ and $g$ is defined by: $(g \circ f)(x) = g(f(x))$  

Note order!
Some Important Functions

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A partial function \( f : A \rightarrow B \) is a total function \( f : A' \rightarrow B \) on a set \( A' \subseteq A \). The notation \( \text{dom}(f) \) refers to \( A' \).
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A function $f : A \to B$ is said to be \textit{surjective} (or \textit{onto}) if and only if the image of $f$ is $B$. 
Operational Semantics
Overview

An **operational semantics** describes how a program executes on some (typically idealized) abstract machine.
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A small-step semantics describes how such an execution proceeds in terms of successive reductions: \( \langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \)
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For our language, a configuration \(\langle \sigma, e \rangle\) has two components:
- a store \(\sigma\) that records the values of variables
- and the expression \(e\) being evaluated
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- a store \( \sigma \) that records the values of variables
- and the expression \( e \) being evaluated

More formally,

\[
\begin{align*}
\text{Store} & \triangleq \text{Var} \rightarrow \text{Int} \\
\text{Config} & \triangleq \text{Store} \times \text{Exp}
\end{align*}
\]

Note that a store is a partial function from variables to integers.
Operational Semantics

The small-step operational semantics itself is a relation on configurations—i.e., a subset of $\text{Config} \times \text{Config}$.
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**Answer:** define it inductively, using inference rules:

\[
p = m + n \\
\frac{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle}{\text{Add}}
\]
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**Question:** How should we define this relation? Note that there are an infinite number of configurations and possible steps!

**Answer:** define it inductively, using inference rules:

\[
p = m + n \\
\frac{p = m + n}{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle} \quad \text{Add}
\]

Intuitively, if facts above the line hold, then facts below the line hold. More formally, "\( \rightarrow \)" is the smallest relation "closed" under the inference rules.
Variables

\[ n = \sigma(x) \]

\[ \langle \sigma, x \rangle \quad \xrightarrow{\text{Var}} \quad \langle \sigma, n \rangle \]
Addition

\[
\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e'_1 + e_2 \rangle}
\]

LAdd

\[p = m + n \langle n + m \rangle \rightarrow \langle p \rangle\]

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Addition

\[
\begin{align*}
\langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e'_1 \rangle & \text{LAdd} \\
\langle \sigma, e_1 + e_2 \rangle & \rightarrow \langle \sigma', e'_1 + e_2 \rangle \\
\langle \sigma, e_2 \rangle & \rightarrow \langle \sigma', e'_2 \rangle & \text{RAdd} \\
\langle \sigma, n + e_2 \rangle & \rightarrow \langle \sigma', n + e'_2 \rangle
\end{align*}
\]
Addition

\[
\frac{\langle \sigma, e_1 \rangle}{\langle \sigma, e_1 + e_2 \rangle} \quad \frac{\langle \sigma', e'_1 \rangle}{\langle \sigma', e'_1 + e_2 \rangle} \quad \text{LAdd}
\]

\[
\frac{\langle \sigma, e_2 \rangle}{\langle \sigma, n + e_2 \rangle} \quad \frac{\langle \sigma', e'_2 \rangle}{\langle \sigma', n + e'_2 \rangle} \quad \text{RAdd}
\]

\[
p = m + n \quad \frac{\langle \sigma, p \rangle}{\langle \sigma, n + m \rangle} \quad \text{Add}
\]
Multiplication

\[
\begin{align*}
\langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e_1' \rangle \\
\langle \sigma, e_1 \ast e_2 \rangle & \rightarrow \langle \sigma', e_1' \ast e_2 \rangle
\end{align*}
\]

\text{LMul}
Multiplication

\[
\begin{align*}
\langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e'_1 \rangle & \text{LMul} \\
\langle \sigma, e_1 \ast e_2 \rangle & \rightarrow \langle \sigma', e'_1 \ast e_2 \rangle \\
\langle \sigma, e_2 \rangle & \rightarrow \langle \sigma', e'_2 \rangle & \text{RMul} \\
\langle \sigma, n \ast e_2 \rangle & \rightarrow \langle \sigma', n \ast e'_2 \rangle
\end{align*}
\]
Multiplication

\[
\frac{\langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 \ast e_2 \rangle \longrightarrow \langle \sigma', e_1' \ast e_2 \rangle} \quad \text{LMul}
\]

\[
\frac{\langle \sigma, e_2 \rangle \longrightarrow \langle \sigma', e_2' \rangle}{\langle \sigma, n \ast e_2 \rangle \longrightarrow \langle \sigma', n \ast e_2' \rangle} \quad \text{RMul}
\]

\[
p = m \times n \\
\frac{\langle \sigma, m \ast n \rangle \longrightarrow \langle \sigma, p \rangle}{\langle \sigma, m \ast n \rangle \longrightarrow \langle \sigma, p \rangle} \quad \text{Mul}
\]
Assignment

\[
\langle \sigma, e_1 \rangle \quad \rightarrow \quad \langle \sigma', e'_1 \rangle \\
\langle \sigma, x := e_1 ; e_2 \rangle \quad \rightarrow \quad \langle \sigma', x := e'_1 ; e_2 \rangle
\]

Assgn1

Notation: \([x \mapsto n] \) maps \(x\) to \(n\) and otherwise behaves like
Assignment

\[
\frac{\langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \longrightarrow \langle \sigma', x := e'_1 ; e_2 \rangle} \quad \text{Assgn1}
\]

\[
\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n ; e_2 \rangle \longrightarrow \langle \sigma', e_2 \rangle} \quad \text{Assgn}
\]

Notation: \( \sigma[x \mapsto n] \) maps \( x \) to \( n \) and otherwise behaves like \( \sigma \)
Operational Semantics

\[ n = \sigma(x) \quad \frac{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle}{\text{Var}} \]

\[ \langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle \quad \frac{\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e'_1 + e_2 \rangle}{\text{LAdd}} \]

\[ \langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle \quad \frac{\langle \sigma, n + e_2 \rangle \rightarrow \langle \sigma', n + e'_2 \rangle}{\text{RAdd}} \]

\[ \langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle \quad \frac{\langle \sigma, e_1 * e_2 \rangle \rightarrow \langle \sigma', e'_1 * e_2 \rangle}{\text{LMul}} \]

\[ \langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle \quad \frac{\langle \sigma, n * e_2 \rangle \rightarrow \langle \sigma', n * e'_2 \rangle}{\text{RMul}} \]

\[ p = m \times n \quad \frac{\langle \sigma, m * n \rangle \rightarrow \langle \sigma, p \rangle}{\text{Mul}} \]

\[ \langle \sigma, x := e_1 ; e_2 \rangle \rightarrow \langle \sigma', x := e'_1 ; e_2 \rangle \quad \frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle}{\text{Assign1}} \]

\[ \sigma' = \sigma[x \mapsto n] \quad \frac{\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle}{\text{Assign}} \]