Announcements

- None!
Concurrency

All of the languages we have seen so far in this course have been sequential, performing one step of computation at a time.

In the next few lectures we will consider languages where multiple threads of execution may be interleaved simultaneously.

These languages can be used to model computations that execute on parallel and distributed architectures.
IMP with Parallel Composition

As a first step, let’s extend IMP with a new a parallel composition command:
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\[
\begin{align*}
    a & ::= x | n | a_1 + a_2 \\
    b & ::= \text{true} | \text{false} | a_1 < a_2 \\
    c & ::= \text{skip} | x ::= a | c_1; c_2 | \text{if } b \text{ then } c_1 \text{ else } c_2 | \text{while } b \text{ do } c \\
        & | c_1 || c_2
\end{align*}
\]
and extend the small-step operational semantics with the following rules for $c_1 \parallel c_2$, which interleave the execution of $c_1$ and $c_2$:

$$\langle \sigma, c_1 \parallel c_2 \rangle \rightarrow \langle \sigma', c'_1 \parallel c'_2 \rangle$$

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\end{align*}
\]
and extend the small-step operational semantics with the following rules for $c_1 || c_2$, which interleave the execution of $c_1$ and $c_2$:

$$
\langle \sigma, c_1 \rangle \rightarrow \langle \sigma', c'_1 \rangle \\
\langle \sigma, c_1 || c_2 \rangle \rightarrow \langle \sigma', c'_1 || c_2 \rangle \\
\langle \sigma, c_2 \rangle \rightarrow \langle \sigma', c'_2 \rangle \\
\langle \sigma, c_1 || c_2 \rangle \rightarrow \langle \sigma', c_1 || c'_2 \rangle \\
\langle \sigma, \text{skip} || \text{skip} \rangle \rightarrow \langle \sigma, \text{skip} \rangle
$$
Operational Semantics

and extend the small-step operational semantics with the following rules for $c_1 \parallel c_2$, which interleave the execution of $c_1$ and $c_2$:

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\langle \sigma, \ c_1 \parallel c_2 \rangle & \rightarrow \langle \sigma', \ c'_1 \parallel c_2 \rangle \\
\langle \sigma, \ c_2 \rangle & \rightarrow \langle \sigma', \ c'_2 \rangle \\
\langle \sigma, \ c_1 \parallel c_2 \rangle & \rightarrow \langle \sigma', \ c_1 \parallel c'_2 \rangle \\
\langle \sigma, \ textbf{skip} \parallel textbf{skip} \rangle & \rightarrow \langle \sigma, \ textbf{skip} \rangle
\end{align*}
\]

Note that the rules for parallel compositions $c_1 \parallel c_2$ allow either sub-command to take a step; two sub-commands can interleave read and write operations involving the same store.
In the 1970s, Tony Hoare, Robin Milner, and others correctly observed that in the future, computers would have multiple computing cores, but each would have its own independent store.

Hoare’s Communicating Sequential Processes were an early and highly-influential language that capture a *message passing* form of concurrency.

Many languages have built on CSP including Milner’s CCS and $\pi$-calculus, Petri nets, and others.
The \( \pi \)-calculus is a minimal formalism that attempts to capture the essence of message-passing concurrency.
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The key constructs are based on the ability to interact by sending and receiving channel names.
\(\pi\)-calculus Syntax

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\[ x, y, z \in \mathcal{N} \]  

Names
\[\pi\text{-calculus Syntax}\]

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\[
x, y, z \in \mathcal{N} \\
\pi ::= \tau \mid \overline{x}(y) \mid x(y) \mid [x = y]\pi
\]

*Names*

*Prefixes*
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Names

\begin{align*}
x, y, z & \in \mathcal{N} \\
\pi & ::= \tau \mid \overline{x}(y) \mid x(y) \mid [x = y] \pi
\end{align*}

Prefixes

\begin{align*}
M, N & ::= 0 \mid \pi.P \mid M + M
\end{align*}

Summations
The $\pi$-calculus is a minimal formalism that attempts to capture the essence of message-passing concurrency.

The key constructs are based on the ability to interact by sending and receiving channel names.

<table>
<thead>
<tr>
<th>$x, y, z \in N$</th>
<th>Names</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi ::= \tau</td>
<td>\overline{x}(y)</td>
</tr>
<tr>
<td>$M, N ::= 0</td>
<td>\pi.P</td>
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<tr>
<td>$P, Q, R ::= M</td>
<td>P_1</td>
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The $\pi$-calculus is a minimal formalism that attempts to capture the essence message-passing concurrency.

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Prefixes

\[
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\]  

Summations

\[
P, Q, R ::= M \mid P_1 \mid P_2 \mid \nu x. P \mid !P
\]  

Processes

In examples, we will often abbreviate $\pi.0$ as $\pi$. 

π-calculus Syntax
Reaction

\[ \tau.P + M \rightarrow P \text{ R-Tau} \]
Reaction

$$\tau. P + M \rightarrow P$$  \hspace{1cm} \text{R-Tau}

$$\left(\overline{x} \langle y \rangle . P_1 + M_1 \right) \mid \left(\overline{x} (z) . P_2 + M_2 \right) \rightarrow P_1 \mid P_2 \{y/z\}$$  \hspace{1cm} \text{R-React}
Reaction

\[
\tau.P + M \rightarrow P \quad \text{R-Tau}
\]

\[
(\overline{x}\langle y \rangle.P_1 + M_1) \mid (x(z).P_2 + M_2) \rightarrow P_1 \mid P_2 \{y/z\} \quad \text{R-React}
\]

\[
P_1 \rightarrow P' \quad P_1 \\
P_1 \mid P_2 \rightarrow P' \mid P_2 \quad \text{R-Par}
\]
Reaction

\[
\tau.P + M \rightarrow P \\
(\bar{x}\langle y \rangle . P_1 + M_1) \mid (x(z). P_2 + M_2) \rightarrow P_1 \mid P_2\{y/z}\]

\[
P_1 \rightarrow P'_1 \\
P_1 \mid P_2 \rightarrow P'_1 \mid P_2
\]

\[
P \rightarrow P' \\
\nu x. P \rightarrow \nu x. P'
\]
Reaction

\[
\begin{align*}
\tau.P + M & \rightarrow P & \text{R-Tau} \\
(\langle x \rangle.P_1 + M_1) & | (x(z).P_2 + M_2) & \rightarrow P_1 | P_2 \{y/z\} & \text{R-React} \\
& \\
& \\
& \\
\end{align*}
\]

\[
\begin{align*}
P_1 & \rightarrow P_1' & \text{R-Par} \\
P_1 | P_2 & \rightarrow P_1' | P_2 \\
& \\
& \\
& \\
\end{align*}
\]

\[
\begin{align*}
P & \rightarrow P' & \text{R-Res} \\
\nu x. P & \rightarrow \nu x. P' \\
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Definition (Congruence)

An equivalence relation $S$ is a congruence if $P S Q$ implies $C[P] S C[Q]$ for every context $C$. 
Structural Congruence

Definition (Structural Congruence)

\[ [x = x] \pi.P \equiv \pi.P \]

\[ M_1 + (M_2 + M_3) \equiv (M_1 + M_2) + M_3 \]

\[ P_1 \mid (P_2 \mid P_3) \equiv (P_1 \mid P_2) \mid P_3 \]

\[ M + 0 \equiv M \]

\[ \nu x. \nu y. P \equiv \nu y. \nu x. P \]

\[ \nu x. P_1 \mid P_2 \equiv P_1 \mid (\nu x. P_2), \text{ if } x \not\in \text{FV}(P_1) \]

\[ !P \equiv P \mid !P \]

\[ M_1 + M_2 \equiv M_2 + M_1 \]

\[ P_1 \mid P_2 \equiv P_2 \mid P_1 \]

\[ P \mid 0 \equiv P \]

\[ \nu x. 0 \equiv 0 \]
Structural Congruence

Theorem (Standard Form)

Each process is structurally congruent to one of the form

$$\nu \vec{x}. \ (M_1 \ | \ldots | M_j | !P_1 | \ldots | !P_k)$$

where each $P_i$ is also in standard form.
Structural Congruence

Theorem (Standard Form)

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\[ \nu \bar{x}. \ (M_1 \ | \ \ldots \ | \ M_j \ | !P_1 \ | \ \ldots \ | !P_k) \]

where each \( P_i \) is also in standard form.

Proof (sketch): repeatedly use \( \alpha \)-conversion and the scope extrusion axiom: \( P \ | \ \nu x. \ Q \equiv \nu x. \ P \ | \ Q \).
Example

\[ a(x). \bar{b}(x) \mid \nu z. (\bar{a}(z)) \]
Example

\[ a(x) + b(x) \mid \nu z. (\bar{a}\langle z \rangle + \bar{b}\langle z \rangle) \]
Example

\( !x(u) \cdot \bar{x}\langle\text{succ } u\rangle \)
Programming in the $\pi$-calculus

Just as with $\lambda$-calculus, we can encode richer data structures and computations using the $\pi$-calculus primitives.
Polyadic $\pi$-Calculus

The send and receive primitives are monadic—they communicate a single name over a given channel. It is often useful to be able to send several names.
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We can try to encode polyadic sends and receives as follows:

$$\bar{x}\langle y_1, \ldots, y_k \rangle.P \triangleq \bar{x}\langle y_1 \rangle.\ldots.\bar{x}\langle y_k \rangle.P$$

$$x(z_1, \ldots, z_k).P \triangleq x(z_1)\ldots.\bar{x}\langle z_k \rangle.P$$

But unfortunately this doesn't work... why?
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Polyadic $\pi$-calculus

To obtain an encoding that works correctly, we can create a fresh name and communicate the values over that channel:

$$\bar{x}\langle y_1, \ldots, y_k \rangle.P \triangleq \nu w. (\bar{x}\langle w \rangle.\bar{w}\langle y_1 \rangle.\ldots.\bar{w}\langle y_k \rangle).P$$
where $w \notin \text{FV}(P)$

$$x(z_1, \ldots, z_k).P \triangleq x(w).w(z_1).\ldots.\bar{w}\langle z_k \rangle.P$$
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where $w \not\in \text{FV}(P)$

$$x(z_1, \ldots, z_k).P \triangleq x(w).w(z_1).\ldots.\bar{w}\langle z_k \rangle.P$$

Using this (adequate) encoding, we will freely use polyadic sends and receives in examples.

$$\overline{(\bar{x}\langle \bar{y} \rangle.P_1 + M_1) | (\bar{x}(\bar{z}).P_2 + M_2) \rightarrow P_1 | P_2\{\bar{y}/\bar{z}\}} \quad \text{R-PolyReact}$$
Encoding Recursion

Idea: Suppose we want to encode $P$ where $A(\vec{x}) \triangleq P_A$.

- Pick a name $a$ to stand for $A$.
- Let $(\|Q\|)$ stand for $Q$ with occurrences of $A(\vec{z})$ replaced by $\bar{a}(\vec{z})$.
- Produce $\nu a. \ ((\|P\|) \ | \ !a(\vec{x}).(\|P_A\|))$
Example: Buffer

Consider a recursive definition of a simple buffer:

\[
B(l, r) \triangleq r(x).C\langle x, l, r \rangle
\]

\[
C(x, l, r) \triangleq l\langle x \rangle.B\langle l, r \rangle
\]
Consider a recursive definition of a simple buffer:

\[
B(l, r) \triangleq r(x).C\langle x, l, r \rangle
\]

\[
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\]

When encoded this becomes

\[
\nu b. \nu c. \left( b\langle l, r \rangle \mid ! b(l, r).r(x).\bar{c}\langle x, l, r \rangle \mid ! c(x, l, r).\bar{l}\langle x \rangle.\bar{b}\langle l, r \rangle \right)
\]