Announcements

- PS 5 due tonight
- PS 6 out today
Normalization

The simply-typed lambda calculus enjoys a remarkable property...

...every well-typed program terminates.
Normalization

The simply-typed lambda calculus enjoys a remarkable property...

...every well-typed program terminates.

We’ll spend this lecture proving this fact.
Simply-Typed Lambda Calculus

Syntax

expressions

\[ e ::= x | \lambda x : \tau. e | e_1 e_2 | () \]

values

\[ \nu ::= \lambda x : \tau. e | () \]

types

\[ \tau ::= \text{unit} | \tau_1 \rightarrow \tau_2 \]
Simply-Typed Lambda Calculus

Syntax

expressions

\[ e ::= x \mid \lambda x : \tau . e \mid e_1 \; e_2 \mid () \]

values

\[ v ::= \lambda x : \tau . e \mid () \]

types

\[ \tau ::= \text{unit} \mid \tau_1 \rightarrow \tau_2 \]

Dynamic Semantics

\[ E ::= [\cdot] \mid E \; e \mid v \; E \]

\[ e \rightarrow e' \quad \frac{E[e] \rightarrow E[e']} \]

\[ (\lambda x : \tau . e) \; v \rightarrow e \{ v/x \} \]
Simply-Typed Lambda Calculus

Static Semantics

\[ \Gamma \vdash () : \text{unit} \]

\[ \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \]

T-Var

\[ \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau. e : \tau \rightarrow \tau'} \]

T-Abs

\[ \frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'} \]

T-App
Supporting Lemmas

Lemma (Inversion)

- If $\Gamma \vdash x : \tau$ then $\Gamma(x) = \tau$
- If $\Gamma \vdash \lambda x : \tau_1. e : \tau$ then $\tau = \tau_1 \rightarrow \tau_2$ and $\Gamma, x : \tau_1 \vdash e : \tau_2$.
- If $\Gamma \vdash e_1 e_2 : \tau$ then $\Gamma \vdash e_1 : \tau' \rightarrow \tau$ and $\Gamma \vdash e_2 ty\tau'$. 
## Supporting Lemmas

### Lemma (Inversion)
- If $\Gamma \vdash x : \tau$ then $\Gamma(x) = \tau$
- If $\Gamma \vdash \lambda x : \tau_1. e : \tau$ then $\tau = \tau_1 \rightarrow \tau_2$ and $\Gamma, x : \tau_1 \vdash e : \tau_2$.
- If $\Gamma \vdash e_1 e_2 : \tau$ then $\Gamma \vdash e_1 : \tau' \rightarrow \tau$ and $\Gamma \vdash e_2 t y \tau'$.

### Lemma (Canonical Forms)
- If $\Gamma \vdash v : \text{unit}$ then $v = ()$
- If $\Gamma \vdash v : \tau_1 \rightarrow \tau_2$ then $v = \lambda x : \tau_1. e$ and $\Gamma, x : \tau_1 \vdash e : \tau_2.$
First Attempt

Theorem (Normalization)

If $\vdash e : \tau$ then there exists a value $v$ such that $e \rightarrow^* v$.

(Proof attempt on board)
Idea: define a set with the following properties:

- At base types the set contains all expressions satisfying some property.
- At function types, the set contains all expressions such that the property is preserved whenever we apply the function to an argument of appropriate type that is also in the set.
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In our setting, the property will concern normalization...
Definition (Logical Relation)

- \( R_{\text{unit}}(e) \) iff \( \vdash e : \text{unit} \) and \( e \) halts.
- \( R_{\tau_1 \rightarrow \tau_2}(e) \) iff \( \vdash e : \tau_1 \rightarrow \tau_2 \) and \( e \) halts, and for every \( e' \) such that \( R_{\tau_1}(e') \) we have \( R_{\tau_2}(e e') \).
Supporting Lemmas

Lemma

If $R_\tau(e)$ then $e$ halts.
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Lemma

If $\vdash e : \tau$ and $e \rightarrow e'$ then $R_{\tau}(e)$ iff $R_{\tau}(e')$. 
Supporting Lemmas

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Lemma

If $\vdash e: \tau$ then $R_\tau(e)$
Main Lemma

Lemma

If

- \( x_1 : \tau_1 \ldots x_k : \tau_k \vdash e : \tau, \)
- \( \nu_1 \) to \( \nu_k \) are values such that \( \vdash \nu_1 : \tau_1 \) to \( \vdash \nu_k : \tau_k \), and
- \( R_{\tau_1}(\nu_1) \) to \( R_{\tau_k}(\nu_k) \),

then \( R_{\tau}(e\{\nu_1/x_1\} \ldots \{\nu_k/x_k\}). \)

(Proof on board)