Announcements

Wednesday Lecture
- Moved to Thurston 203

Foster Office Hours
- Today 11a-12pm in Gates 432

Mota Office Hours
- Wed 11am-12pm in TBD
- Thurs 2:30pm-4pm in TBD

Homework #1
- Out: Wednesday, September 3rd
- Due: Wednesday, September 10th
- Distributed via CMS
Question: What is the meaning of a program?
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Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...

...but none of these is a satisfactory solution.
Formal Semantics

Three Approaches

• Operational
  ▶ Model program by execution on abstract machine
  ▶ Useful for implementing compilers and interpreters

• Denotational:
  ▶ Model program as mathematical objects
  ▶ Useful for theoretical foundations

• Axiomatic
  ▶ Model program by the logical formulas it obeys
  ▶ Useful for proving program correctness
Arithmetic Expressions
Syntax

A language of integer arithmetic expressions with assignment.
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Metavariables:

\[ x, y, z \in \text{Var} \]
\[ n, m \in \text{Int} \]
\[ e \in \text{Exp} \]
Syntax

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Metavariabes:

\[ \begin{align*}
  x, y, z & \in \text{Var} \\
  n, m & \in \text{Int} \\
  e & \in \text{Exp}
\end{align*} \]

BNF Grammar:

\[ e ::= x \\
    n \\
    e_1 + e_2 \\
    e_1 \times e_2 \\
    x := e_1 ; e_2 \]
What expression does the string “1 + 2 * 3” describe?
Ambiguity

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There are two possible parse trees:
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In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).
Representing Expressions

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Representing Expressions

BNF Grammar:

\[ e ::= x \]
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\[ | e_1 * e_2 \]
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OCaml:

```ocaml
type exp = Var of string
| Int of int
| Add of exp * exp
| Mul of exp * exp
| Assgn of string * exp * exp
```

Example: \(\text{Mul} (\text{Int} 2, \text{Add} (\text{Var} "\text{foo}", \text{Int} 1))\)
Representing Expressions

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| n \\
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\]

Java:

abstract class Expr {}
class Var extends Expr { String name; ... }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assgn extends Expr { String var, Expr exp1, exp2; ... }

Example: new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))
Quiz

- $7 + (4 \times 2)$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15

The rest of this lecture will make these intuitions precise...
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1; 2 \times 3 \times i$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
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Quiz

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- $i := 6 + 1; 2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to ...?
Quiz

- $7 + (4 \times 2)$ evaluates to 15
- $i := 6 + 1$; $2 \times 3 \times i$ evaluates to 42
- $x + 1$ evaluates to error?
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- $x + 1$ evaluates to error?

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Mathematical Preliminaries
Binary Relations

The *product* of two sets $A$ and $B$, written $A \times B$, contains all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$. 
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Some Important Relations

- empty – $\emptyset$
- total – $A \times B$
- identity on $A$ – $\{(a, a) \mid a \in A\}$.
- composition $R; S$ – $\{(a, c) \mid \exists b. (a, b) \in R \land (b, c) \in S\}$
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When $f$ is a function, we usually write $f : A \rightarrow B$ instead of $f \subseteq A \times B$.
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The \textit{domain} and \textit{range} of $f$ are defined the same way as for relations.
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The image of $f$ is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. More formally: $\text{image}(f) \triangleq \{ f(a) \mid a \in A \}$
Some Important Functions

Given two functions $f : A \to B$ and $g : B \to C$, the composition of $f$ and $g$ is defined by: $(g \circ f)(x) = g(f(x))$ Note order!
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A function $f : A \to B$ is said to be injective (or one-to-one) if and only if $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. 
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A function $f : A \to B$ is said to be surjective (or onto) if and only if the image of $f$ is $B$. 
Operational Semantics
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For our language, a configuration \( \langle \sigma, e \rangle \) has two components:

- a store \( \sigma \) that records the values of variables
- and the expression \( e \) being evaluated
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- and the expression \( e \) being evaluated

More formally,

\[
\begin{align*}
\text{Store} & \triangleq \text{Var} \rightarrow \text{Int} \\
\text{Config} & \triangleq \text{Store} \times \text{Exp}
\end{align*}
\]

Note that a store is a partial function from variables to integers.
The small-step operational semantics itself is a relation on configurations—i.e., a subset of $\text{Config} \times \text{Config}$. 

**Notation:**

$$\langle ; e \rangle ! \langle ; e' \rangle$$

**Question:** How should we define this relation?

**Answer:** define it inductively, using inference rules:

$$p = m + n \quad \langle ; n + m \rangle ! \langle ; p \rangle$$

Intuitively, if facts above the line hold, then facts below the line hold. More formally, "!" is the smallest relation "closed" under the inference rules.
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$$\frac{p = m + n}{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle} \text{ Add}$$
Operational Semantics

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**Answer:** define it inductively, using inference rules:

\[
\frac{p = m + n}{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle} \quad \text{Add}
\]

Intuitively, if facts above the line hold, then facts below the line hold. More formally, “\( \rightarrow \)” is the smallest relation “closed” under the inference rules.
Variables

\[
n = \sigma(x) \\
\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle \quad \text{Var}
\]
Addition

\[
\begin{align*}
\langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e'_1 \rangle \\
\langle \sigma, e_1 + e_2 \rangle & \rightarrow \langle \sigma', e'_1 + e_2 \rangle
\end{align*}
\]

LAdd
Addition

\[
\begin{align*}
\langle \sigma, e_1 \rangle &\quad \rightarrow \quad \langle \sigma', e'_1 \rangle & \text{LAdd} \\
\langle \sigma, e_1 + e_2 \rangle &\quad \rightarrow \quad \langle \sigma', e'_1 + e_2 \rangle \\
\langle \sigma, e_2 \rangle &\quad \rightarrow \quad \langle \sigma', e'_2 \rangle & \text{RAdd} \\
\langle \sigma, n + e_2 \rangle &\quad \rightarrow \quad \langle \sigma', n + e'_2 \rangle
\end{align*}
\]
Addition

\[
\begin{align*}
\langle \sigma, e_1 \rangle &\xrightarrow{\text{LAdd}} \langle \sigma', e'_1 \rangle \\
\langle \sigma, e_1 + e_2 \rangle &\xrightarrow{\text{LAdd}} \langle \sigma', e'_1 + e_2 \rangle \\
\langle \sigma, e_2 \rangle &\xrightarrow{\text{RAdd}} \langle \sigma', e'_2 \rangle \\
\langle \sigma, n + e_2 \rangle &\xrightarrow{\text{RAdd}} \langle \sigma', n + e'_2 \rangle \\
p = m + n &\xrightarrow{\text{Add}} \langle \sigma, p \rangle
\end{align*}
\]
Multiplication

\[ \langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e'_1 \rangle \quad \text{LMul} \]

\[ \langle \sigma, e_1 \star e_2 \rangle \longrightarrow \langle \sigma', e'_1 \star e_2 \rangle \]
Multiplication

\[
\begin{align*}
\langle \sigma, e_1 \rangle & \rightarrow \langle \sigma', e'_1 \rangle & \text{LMul} \\
\langle \sigma, e_1 * e_2 \rangle & \rightarrow \langle \sigma', e'_1 * e_2 \rangle \\
\langle \sigma, e_2 \rangle & \rightarrow \langle \sigma', e'_2 \rangle & \text{RMul} \\
\langle \sigma, n * e_2 \rangle & \rightarrow \langle \sigma', n * e'_2 \rangle
\end{align*}
\]
Multiplication

\[
\frac{\langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, e_1 \times e_2 \rangle \longrightarrow \langle \sigma', e'_1 \times e_2 \rangle} \quad \text{LMul}
\]

\[
\frac{\langle \sigma, e_2 \rangle \longrightarrow \langle \sigma', e'_2 \rangle}{\langle \sigma, n \times e_2 \rangle \longrightarrow \langle \sigma', n \times e'_2 \rangle} \quad \text{RMul}
\]

\[
p = m \times n
\]

\[
\frac{p = m \times n}{\langle \sigma, m \times n \rangle \longrightarrow \langle \sigma, p \rangle} \quad \text{Mul}
\]
Assignment

\[
\begin{align*}
\langle \sigma, e_1 \rangle & \rightsquigarrow \langle \sigma', e'_1 \rangle \\
\langle \sigma, x := e_1 ; e_2 \rangle & \rightsquigarrow \langle \sigma', x := e'_1 ; e_2 \rangle
\end{align*}
\]

Assgn1

Notation: \([x := n]\) maps \(x\) to \(n\) and otherwise behaves like
Assignment

\[
\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle
\]

\[
\langle \sigma, x := e_1 ; e_2 \rangle \rightarrow \langle \sigma', x := e'_1 ; e_2 \rangle
\]

\[
\sigma' = \sigma[x \mapsto n]
\]

\[
\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle
\]

Notation: \( \sigma[x \mapsto n] \) maps \( x \) to \( n \) and otherwise behaves like \( \sigma \)
Operational Semantics

\[
\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle} \quad \text{Var}
\]

\[
\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e_1' + e_2 \rangle} \quad \text{LAdd}
\]

\[
\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e_2' \rangle}{\langle \sigma, n + e_2 \rangle \rightarrow \langle \sigma', n + e_2' \rangle} \quad \text{RAdd}
\]

\[
\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 \ast e_2 \rangle \rightarrow \langle \sigma', e_1' \ast e_2 \rangle} \quad \text{LMul}
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\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e_2' \rangle}{\langle \sigma, n \ast e_2 \rangle \rightarrow \langle \sigma', n \ast e_2' \rangle} \quad \text{RMul}
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\[
\frac{p = m \times n}{\langle \sigma, m \ast n \rangle \rightarrow \langle \sigma, p \rangle} \quad \text{Mul}
\]

\[
\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \rightarrow \langle \sigma', x := e_1' ; e_2 \rangle} \quad \text{Assgn1}
\]

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\sigma' = \sigma[x \mapsto n] \quad \text{Assgn}
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\frac{\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle}{\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle}
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