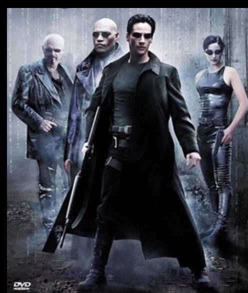
Linear Systems



What is the Matrix?

Outline

- Announcements:
  - Homework III: due Wed. by 5, by e-mail
    - Office Hours: Today & tomorrow, 11-1
  - Ideas for Friday?
- Linear Systems Basics
- Matlab and Linear Algebra

Ecology of Linear Systems

- Linear Systems are found in every habitat:
  - simple algebra
  - solutions to ODEs & PDEs
  - statistics (especially, least squares)
- If you can formulate your problem as linear system, it will be easy to solve on a computer

"Standard Linear System"

- Simplest linear system: finding the equation of a line:
  - y=m*x + b
- The goal is to find m and b from observations of (x,y) pairs
- 2 points form a line, so we need two observations (x1, y1) & (x2,y2)

$$\begin{aligned} y_1 &= mx_1 + b \\ y_2 &= mx_2 + b \end{aligned}$$

The 8th Grade Solution

- Solve for b in the first equation:

$$b = y_1 - mx_1$$

- Substitute this for b in 2nd equation & solve for m:

$$\begin{aligned} y_2 &= mx_2 + y_1 - mx_1 \\ m &= \frac{y_2 - y_1}{x_2 - x_1} \end{aligned}$$

- Put this into the equation for b above

The Sophomoric Solution

- Write the equations as a matrix problem

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} * \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Perform Gaussian Elimination on Matrix:

$$\left[\begin{array}{cc|c} 1 & 1/x_1 & y_1/x_1 \\ x_2 & 1 & y_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1/x_1 & y_1/x_1 \\ 0 & \frac{x_1-x_2}{x_1} & \frac{x_1y_2-x_2y_1}{x_1} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1/x_1 & y_1/x_1 \\ 0 & 1 & \frac{x_1y_2-x_2y_1}{x_1-x_2} \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{y_1-y_2}{x_1-x_2} \\ 0 & \frac{x_1-x_2}{x_1} & \frac{x_1y_2-x_2y_1}{x_1-x_2} \end{array}\right]$$

Comparing methods

- Gaussian Elimination is a simpler algorithm
  - Easily generalizes to systems with more unknowns
- Gaussian Elimination is the starting point of much of numerical linear algebra

A Closer Look at GE (optional)

- For Am=y
  - GE reduces A to an upper triangular matrix
  - Perform "back substitution" using modified y
  - Modified y is equivalent to Ly
    - L is a lower triangular matrix
    - A=L*U

Other Algorithms for Solving Linear Systems

- GE aka LU decomposition -- any A
- Cholesky Factorization -- symmetric, positive definite A
- Iterative solvers (conjugate gradients, GMRES)

Linear Systems in Matlab

- Linear systems are easy in Matlab
  - To solve Ax=b, type x=A\b
  - To solve x'A'=b', type x'=b'/A' (transposed)

More About \

- Matrix multiplication (*) is easy, fast
- Matrix "division" (\) is hard and computationally intensive
  - In general, performs GE with partial pivoting
  - But, \ is smart & looks closely at A for opportunities to speed up
    - If A is LT, just does back substitution
- If A is over-determined, A\b is the least-squares solution

Factorization

- Can explicitly factor A using LU:
  - [L,U]=lu(A)
  - useful if you have to solve A\bmany times (different b each time)
    - To solve LUx=b: first solve Ly=b, then solve Ux=y
    - In Matlab: y=L\b; x=U\y;
- Other factorizations: chol, svd

What about A^{-1}?

- Matlab can compute A^{-1} using inv(A), but …
  - inv(A) is slower than lu(A)
  - There are numerical problems with inv(A)
- Rarely needed, use lu(A) or another factorization

$$
\begin{aligned}
s &= c' A^{-1} d \\
s &= c' U^{-1} L^{-1} d \\
Ly &= d \\
Ux &= y \\
s &= c' x
\end{aligned}
$$