



PROPOSITIONAL CALCULUS **E**

INFERENCE RULES

$$(3.1) \text{ Leibniz: } \frac{P = Q}{E[V := P] = E[V := Q]} \quad (V \text{ a metavariable})$$

$$(3.2) \text{ Transitivity: } \frac{P = Q, Q = R}{P = R}$$

$$(3.3) \text{ Equanimity: } \frac{P, P \equiv Q}{Q}$$

DERIVED INFERENCE RULES

$$(3.11) \text{ Redundant } true: \frac{P}{P \equiv true} \quad \frac{P \equiv true}{P}$$

$$(3.91) \text{ Modus ponens: } \frac{P, P \Rightarrow Q}{Q}$$

$$(4.8) \text{ Monotonicity: } \frac{P \Rightarrow Q}{E[V := P] \Rightarrow E[V := Q]} \quad (\text{for parity of } V \text{ even in } E)$$

$$(4.9) \text{ Antimonotonicity: } \frac{P \Rightarrow Q}{E[V := P] \Leftarrow E[V := Q]} \quad (\text{for parity of } V \text{ odd in } E)$$

EQUIVALENCE AND TRUE

$$(3.5) \text{ Axiom, Associativity of } \equiv: ((P \equiv Q) \equiv R) \equiv (P \equiv (Q \equiv R))$$

$$(3.6) \text{ Axiom, Symmetry of } \equiv: P \equiv Q \equiv Q \equiv P$$

$$(3.7) \text{ Axiom, Identity of } \equiv: true \equiv Q \equiv Q$$

$$(3.8) \text{ Reflexivity of } \equiv: P \equiv P$$

$$(3.9) \text{ true}$$

$$(3.12) \text{ Metatheorem: Any two theorems are equivalent.}$$

NEGATION, INEQUIVALENCE, AND FALSE

$$(3.15) \text{ Axiom, Definition of } false: false \equiv \neg true$$

$$(3.17) \text{ Axiom, Distributivity of } \neg \text{ over } \equiv: \neg(P \equiv Q) \equiv \neg P \equiv Q$$

$$(3.14) \text{ Axiom, Definition of } \neq: (P \neq Q) \equiv \neg(P \equiv Q)$$

$$(3.18) \neg P \equiv Q \equiv P \equiv \neg Q$$

$$(3.19) \text{ Double negation: } \neg\neg P \equiv P$$

$$(3.16) \text{ Negation of } false: \neg false \equiv true$$

$$(3.20) (P \neq Q) \equiv \neg P \equiv Q$$

$$(3.13) \neg P \equiv P \equiv false$$

$$(3.21) \text{ Symmetry of } \neq: (P \neq Q) \equiv (Q \neq P)$$

$$(3.22) \text{ Associativity of } \neq: ((P \neq Q) \neq R) \equiv (P \neq (Q \neq R))$$

$$(3.23) \text{ Mutual associativity: } ((P \neq Q) \equiv R) \equiv (P \neq (Q \equiv R))$$



(3.24) **Mutual interchangeability:** $P \neq Q \equiv R \equiv P \equiv Q \neq R$

DISJUNCTION

(3.29) **Axiom, Symmetry of \vee :** $P \vee Q \equiv Q \vee P$

(3.30) **Axiom, Associativity of \vee :** $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

(3.31) **Axiom, Idempotency of \vee :** $P \vee P \equiv P$

(3.32) **Axiom, Distributivity of \vee over \equiv :**
 $P \vee (Q \equiv R) \equiv P \vee Q \equiv P \vee R$

(3.33) **Axiom, Excluded Middle:** $P \vee \neg P \equiv \text{true}$

(3.34) **Zero of \vee :** $P \vee \text{true} \equiv \text{true}$

(3.35) **Identity of \vee :** $P \vee \text{false} \equiv P$

(3.36) **Distributivity of \vee over \vee :** $P \vee (Q \vee R) \equiv (P \vee Q) \vee (P \vee R)$

(3.37) $P \vee Q \equiv P \vee \neg Q \equiv P$

CONJUNCTION

(3.40) **Axiom, Golden rule:** $P \wedge Q \equiv P \equiv Q \equiv P \vee Q$

(3.41) **Symmetry of \wedge :** $P \wedge Q \equiv Q \wedge P$

(3.42) **Associativity of \wedge :** $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

(3.43) **Idempotency of \wedge :** $P \wedge P \equiv P$

(3.44) **Identity of \wedge :** $P \wedge \text{true} \equiv P$

(3.45) **Zero of \wedge :** $P \wedge \text{false} \equiv \text{false}$

(3.46) **Distributivity of \wedge over \wedge :** $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge (P \wedge R)$

(3.47) **Contradiction:** $P \wedge \neg P \equiv \text{false}$

(3.48) **Absorption:** (a) $P \wedge (P \vee Q) \equiv P$

(b) $P \vee (P \wedge Q) \equiv P$

(3.49) **Absorption:** (a) $P \wedge (\neg P \vee Q) \equiv P \wedge Q$

(b) $P \vee (\neg P \wedge Q) \equiv P \vee Q$

(3.50) **Distributivity of \vee over \wedge :** $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

(3.51) **Distributivity of \wedge over \vee :** $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

(3.52) **De Morgan:** (a) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

(b) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

(3.53) $P \wedge Q \equiv P \wedge \neg Q \equiv \neg P$

(3.54) $P \wedge (Q \equiv R) \equiv P \wedge Q \equiv P \wedge R \equiv P$

(3.55) $P \wedge (Q \equiv P) \equiv P \wedge Q$

(3.56) **Replacement:** $(P \equiv Q) \wedge (R \equiv P) \equiv (P \equiv Q) \wedge (R \equiv Q)$

(3.57) **Definition of \equiv :** $P \equiv Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$

(3.58) **Exclusive or:** $P \neq Q \equiv (\neg P \wedge Q) \vee (P \wedge \neg Q)$



$$(3.62) (P \wedge Q) \wedge R \equiv P \equiv Q \equiv R \equiv P \vee Q \equiv Q \vee R \equiv R \vee P \equiv P \vee Q \vee R$$

IMPLICATION

$$(3.65) \text{ Axiom, Implication: } P \Rightarrow Q \equiv P \vee Q \equiv Q$$

$$(3.64) \text{ Axiom, Consequence: } P \Leftarrow Q \equiv Q \Rightarrow P$$

$$(3.66) \text{ Implication: } P \Rightarrow Q \equiv P \wedge Q \equiv P$$

$$(3.63) \text{ Implication: } P \Rightarrow Q \equiv \neg P \vee Q$$

$$(3.67) \text{ Contrapositive: } P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$$

$$(3.68) P \Rightarrow (Q \equiv R) \equiv P \wedge Q \equiv P \wedge R$$

$$(3.69) \text{ Distributivity of } \Rightarrow$$

$$(a) \text{ over } \equiv: P \Rightarrow (Q \equiv R) \equiv P \Rightarrow Q \equiv P \Rightarrow R$$

$$(b) \text{ over } \vee: P \Rightarrow Q \vee R \equiv (P \Rightarrow Q) \vee (P \Rightarrow R)$$

$$(c) \text{ over } \wedge: P \Rightarrow (Q \wedge R) \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R)$$

$$(d) \text{ over } \Rightarrow: P \Rightarrow (Q \Rightarrow R) \equiv (P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$$

$$(3.70) \text{ Shunting: } P \wedge Q \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R)$$

$$(3.71) P \Rightarrow (Q \Rightarrow P)$$

$$(3.72) P \wedge (P \Rightarrow Q) \equiv P \wedge Q$$

$$(3.73) P \wedge (Q \Rightarrow P) \equiv P$$

$$(3.74) P \vee (P \Rightarrow Q) \equiv \text{true}$$

$$(3.75) P \vee (Q \Rightarrow P) \equiv Q \Rightarrow P$$

$$(3.76) P \vee Q \Rightarrow P \wedge Q \equiv P \equiv Q$$

$$(3.77) \text{ Reflexivity of } \Rightarrow: P \Rightarrow P \equiv \text{true}$$

$$(3.78) \text{ Right zero of } \Rightarrow: P \Rightarrow \text{true} \equiv \text{true}$$

$$(3.79) \text{ Left identity of } \Rightarrow: \text{true} \Rightarrow P \equiv P$$

$$(3.80) P \Rightarrow \text{false} \equiv \neg P$$

$$(3.81) \text{false} \Rightarrow P \equiv \text{true}$$

$$(3.82) \text{ Weakening/strengthening: (a) } P \Rightarrow P \vee Q$$

$$(b) P \wedge Q \Rightarrow P$$

$$(c) P \wedge Q \Rightarrow P \vee Q$$

$$(d) P \vee (Q \wedge R) \Rightarrow P \vee Q$$

$$(e) P \wedge Q \Rightarrow P \wedge (Q \vee R)$$

$$(3.83) \text{ Modus ponens: } P \wedge (P \Rightarrow Q) \Rightarrow Q$$

$$(3.84) (P \Rightarrow R) \wedge (Q \Rightarrow R) \equiv (P \vee Q \Rightarrow R)$$

$$(3.85) (P \Rightarrow R) \wedge (\neg P \Rightarrow R) \equiv R$$

$$(3.86) \text{ Mutual implication: } (P \Rightarrow Q) \wedge (Q \Rightarrow P) \equiv (P \equiv Q)$$

$$(3.87) \text{ Antisymmetry: } (P \Rightarrow Q) \wedge (Q \Rightarrow P) \Rightarrow (P \equiv Q)$$

$$(3.88) \text{ Transitivity of } \equiv: (P \equiv Q) \wedge (Q \equiv R) \Rightarrow (P \equiv R)$$

$$(3.89) \text{ Transitivity: (a) } (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$$



$$(b) (P \equiv Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$$

$$(c) (P \Rightarrow Q) \wedge (Q \equiv R) \Rightarrow (P \Rightarrow R)$$

(??) **Metatheorem Modus ponens:**

If P and $P \Rightarrow Q$ are theorems, so is Q

LEIBNIZ AS AN AXIOM

$$(3.93) \text{ **Axiom, Leibniz: } X = Y \Rightarrow E[V:=X] = E[V:=Y]**$$

$$(3.94) \text{ **Substitution: (a) } X = Y \wedge E[V:=X] \equiv X = Y \wedge E[V:=Y]**$$

$$(b) X = Y \Rightarrow E[V:=X] \equiv X = Y \Rightarrow E[V:=Y]$$

$$(c) Q \wedge X = Y \Rightarrow E[V:=X] \equiv Q \wedge X = Y \Rightarrow E[V:=Y]$$

$$(3.95) \text{ **Replace by true: (a) } P \Rightarrow E[V:=P] \equiv P \Rightarrow E[V:=true]**$$

$$(b) Q \wedge P \Rightarrow E[V:=P] \equiv Q \wedge P \Rightarrow E[V:=true]$$

$$(3.96) \text{ **Replace by false: (a) } E[V:=P] \Rightarrow P \equiv E[V:=false] \Rightarrow P**$$

$$(b) E[V:=P] \Rightarrow P \vee Q \equiv E[V:=false] \Rightarrow P \vee Q$$

$$(3.97) \text{ **Replace by true: } P \wedge E[V:=P] \equiv P \wedge E[V:=true]**$$

$$(3.98) \text{ **Replace by false: } P \vee E[V:=P] \equiv P \vee E[V:=false]**$$

$$(3.99) \text{ **Boole: } E[V:=P] \equiv (P \wedge E[V:=true]) \vee (\neg P \wedge E[V:=false])**$$

MONOTONICITY THEOREMS

$$(4.1) \text{ **Monotonic } \vee: (P \Rightarrow Q) \Rightarrow (P \vee R \Rightarrow Q \vee R)**$$

$$(4.2) \text{ **Monotonic } \wedge: (P \Rightarrow Q) \Rightarrow (P \wedge R \Rightarrow Q \wedge R)**$$

$$(4.3) \text{ **Monotonic consequent: } (P \Rightarrow Q) \Rightarrow ((R \Rightarrow P) \Rightarrow (R \Rightarrow Q))**$$

$$(4.4) \text{ **Antimonotonic } \neg: (P \Rightarrow Q) \Rightarrow (\neg P \Leftarrow \neg Q)**$$

$$(4.5) \text{ **Antimonotonic antecedent: } (P \Rightarrow Q) \Rightarrow ((P \Rightarrow R) \Leftarrow (Q \Rightarrow R))**$$

(4.7) **Metatheorem Monotonicity:**

Provided $P \Rightarrow Q$ is a theorem and the parity of the position of V in E is even, $E[V:=P] \Rightarrow E[V:=Q]$.

Provided $P \Rightarrow Q$ is a theorem and the parity of the position of V in E is odd, $E[V:=P] \Leftarrow E[V:=Q]$.

$$(4.11) (P \Rightarrow P') \Rightarrow ((Q \Rightarrow Q') \Rightarrow (P \wedge Q \Rightarrow P' \wedge Q'))$$

$$(4.12) (P \vee Q \vee R) \wedge (P \Rightarrow S) \wedge (Q \Rightarrow S) \wedge (R \Rightarrow S) \Rightarrow S$$

PROOF TECHNIQUES AND STRATEGIES

(3.10) **Basic strategies:** To prove $P \equiv Q$, transform P to Q , transform Q to P , or transform $P \equiv Q$ to a previous theorem.

(3.26) **Heuristic:** Identify applicable theorems by matching the structure of subexpressions with the theorems.

(3.27) **Principle:** Structure proofs to avoid repeating the same subexpression on many lines.



- (3.28) **Heuristic Unfold-fold:** To prove a theorem concerning an operator, remove it using its definition, manipulate, and reintroduce it using its definition.
- (3.38) **Heuristic:** To prove $P \equiv Q$, transform the expression with the most structure (either P or Q) into the other.
- (3.39) **Principle:** Structure proofs to minimize the number of rabbits.
- (3.61) **Principle:** Lemmas can provide structure, bring to light interesting facts, and ultimately shorten proofs.
- (3.60) **Heuristic:** Exploit the ability to parse theorems like the Golden rule in many different ways.
- (4.10) **Deduction:** To prove $P \Rightarrow Q$, assume P and prove Q , with the variables and metavariables of P treated as constants.
- (4.12) **Case analysis:** To prove S , prove $P \vee Q \vee R$, $P \Rightarrow S$, $Q \Rightarrow S$, and $R \Rightarrow S$.
- (4.13) **Partial evaluation:** To prove $E[V := P]$, prove $E[V := true]$ and $E[V := false]$.
- (4.14) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.
- (4.19) **Contradiction:** To prove P , prove $\neg P \Rightarrow false$ (perhaps by assuming $\neg P$ and proving $false$).
- (4.22) **Contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$.
- (5.2) **Heuristic:** To show that an English argument is sound, formalize it and prove that its formalization is a theorem.
- (5.3) **Heuristic:** To disprove an expression (prove that it is not a theorem), find a counterexample: a state in which the expression is *false*.

GENERAL LAWS OF QUANTIFICATION

For symmetric and associative binary operator \star with identity \mathbf{u} .

$$(8.18) \text{ Leibniz: } \frac{R \Rightarrow P = Q}{(\star x \mid R : E[V := P]) = (\star x \mid R : E[V := Q])}$$

- (8.19) **Caveat.** The axioms and theorems for all quantifications hold only in cases in which all the individual quantifications are defined or specified.

THEOREMS

- (8.20) **Axiom, Empty range:** $(\star x \mid false : P) = \mathbf{u}$
- (8.21) **Axiom, Identity accumulation:** $(\star x \mid R : \mathbf{u}) = \mathbf{u}$
- (8.22) **Axiom, One-point rule:** Provided $\neg occurs('x', 'E')$,
 $(\star x \mid x = E : P) = P_E^x$
- (8.23) **Axiom, Distributivity:**
 $(\star x \mid R : P) \star (\star x \mid R : Q) = (\star x \mid R : P \star Q)$



- (8.24) **Axiom, Range split:**
 $(\star x \mid R \vee S : P) \star (\star x \mid R \wedge S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$
- (8.25) **Axiom, Interchange of dummies:** Provided $\neg occurs('y', 'R')$
and $\neg occurs('x', 'Q')$,
 $(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$
- (8.26) **Axiom, Nesting:** Provided $\neg occurs('y', 'R')$,
 $(\star x, y \mid R \wedge Q : P) = (\star x \mid R : (\star y \mid Q : P))$
- (8.27) **Range split:** Provided $R \wedge S \equiv false$
 $(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$
- (8.28) **Range split for idempotent \star :**
 $(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$
- (8.29) **Dummy reordering:** $(\star x, y \mid R : P) = (\star y, x \mid R : P)$
- (8.30) **Change of dummy:** Provided $\neg occurs('y', 'R, P')$, and f
has an inverse,
 $(\star x \mid R : P) = (\star y \mid R_{f,y}^x : P_{f,y}^x)$
- (8.31) **Dummy renaming:** Provided $\neg occurs('y', 'R, P')$,
 $(\star x \mid R : P) = (\star y \mid R_y^x : P_y^x)$
- (8.32) **Split off term (example):**
 $(\star i \mid 0 \leq i < n + 1 : P) = (\star i \mid 0 \leq i < n : P) \star P_n^i$

THEOREMS OF THE PREDICATE CALCULUS

UNIVERSAL QUANTIFICATION

- (9.3) **Axiom, Distributivity of \vee over \forall :** Provided $\neg occurs('x', 'P')$,
 $P \vee (\forall x \mid R : Q) \equiv (\forall x \mid R : P \vee Q)$
- (9.4) **Trading:** (a) $(\forall x \mid R : P) \equiv (\forall x \mid R : P)$
(b) $(\forall x \mid R : P) \equiv (\forall x \mid R \vee P \equiv P)$
(c) $(\forall x \mid R : P) \equiv (\forall x \mid R \wedge P \equiv R)$
(d) $(\forall x \mid R : P) \equiv (\forall x \mid R : \neg R \vee P)$
- (9.5) **Trading:** (a) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$
(b) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \vee P \equiv P)$
(c) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \wedge P \equiv R)$
(d) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : \neg R \vee P)$
- (9.6) **\forall -True body:** $(\forall x \mid R : true) \equiv true$
- (9.7) **\forall -False body:** $(\forall x \mid R : false) \equiv false$
- (9.8) **\forall -Split-off-term:** $(\forall x \mid R : P) \equiv (\forall x \mid R : P) \wedge P_E^x$
- (9.9) **Quantified freshy:** Provided $\neg occurs('x', 'P')$,
 $P \Rightarrow (\forall x \mid R : P)$
- (9.10) Provided $\neg occurs('x', 'P')$,
 $P \wedge (\forall x \mid R : Q) \Rightarrow (\forall x \mid R : P \wedge Q)$
- (9.11) **Distributivity of \wedge over \forall :** Provided $\neg occurs('x', 'P')$,
 $\neg(\forall x \mid R : \neg R) \Rightarrow ((\forall x \mid R : P \wedge Q) \equiv P \wedge (\forall x \mid R : Q))$
- (9.12) $(\forall x \mid R : P \equiv Q) \Rightarrow ((\forall x \mid R : P) \equiv (\forall x \mid R : Q))$



- (9.13) Provided $\neg\text{occurs}('x', 'Q')$,
 $Q \Rightarrow (\forall x \mid R : P) \equiv (\forall x \mid R : Q \Rightarrow P)$
- (9.14) **Monotonic \forall -body:**
 $(\forall x \mid R : P \Rightarrow Q) \Rightarrow ((\forall x \mid R : P) \Rightarrow (\forall x \mid R : Q))$
- (9.15) **Antimonotonic \forall -range:**
 $(\forall x \mid \neg R : P \Rightarrow Q) \Rightarrow ((\forall x \mid P : R) \Leftarrow (\forall x \mid Q : R))$
- (9.17) **Instantiation:** $(\forall x \mid : P) \Rightarrow P_E^x$
- (9.18) Provided $\neg\text{occurs}('x', 'P')$, $(\forall x \mid : P) \equiv P$
- (9.21) **Metatheorem:** P is a theorem iff $(\forall x \mid : P)$ is a theorem.

EXISTENTIAL QUANTIFICATION

- (9.22) **Axiom, Generalized De Morgan:**
 $(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$
- (9.23) **Generalized De Morgan:** (a) $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$
(b) $\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$
(c) $(\exists x \mid R : \neg P) \equiv \neg(\forall x \mid R : P)$
- (9.25) **Distributivity of \wedge over \exists :** Provided $\neg\text{occurs}('x', 'P')$,
 $P \wedge (\exists x \mid R : Q) \equiv (\exists x \mid R : P \wedge Q)$
- (9.26) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$
- (9.27) **Trading:** $(\exists x \mid Q \wedge R : P) \equiv (\exists x \mid Q : R \wedge P)$
- (9.28) **\exists -False body:** $(\exists x \mid R : \text{false}) \equiv \text{false}$
- (9.29) **\exists -True body:** $(\exists x \mid : \text{true}) \equiv \text{true}$
- (9.30) **\exists -Split-off-term:** $(\exists x \mid : P) \equiv (\exists x \mid : P) \vee P_E^x$
- (9.31) **Quantified freshy:** Provided $\neg\text{occurs}('x', 'P')$,
 $(\exists x \mid R : P) \Rightarrow P$
- (9.32) Provided $\neg\text{occurs}('x', 'P')$,
 $(\exists x \mid R : P \vee Q) \Rightarrow P \vee (\exists x \mid R : Q)$
- (9.33) **Distributivity of \vee over \exists :** Provided $\neg\text{occurs}('x', 'P')$,
 $(\exists x \mid : R) \Rightarrow ((\exists x \mid R : P \vee Q) \equiv P \vee (\exists x \mid R : Q))$
- (9.34) $(\forall x \mid R : P \equiv Q) \Rightarrow ((\exists x \mid R : P) \equiv (\exists x \mid R : Q))$
- (9.35) Provided $\neg\text{occurs}('x', 'Q')$,
 $(\exists x \mid R : P) \Rightarrow Q \equiv (\forall x \mid R : P \Rightarrow Q)$
- (9.36) **Monotonic \exists -body:**
 $(\forall x \mid R : P \Rightarrow Q) \Rightarrow ((\exists x \mid R : P) \Rightarrow (\exists x \mid R : Q))$
- (9.37) **Monotonic \exists -range:**
 $(\forall x \mid R : P \Rightarrow Q) \Rightarrow ((\exists x \mid P : R) \Rightarrow (\exists x \mid Q : R))$
- (9.38) **\exists -Introduction:** $P_E^x \Rightarrow (\exists x \mid : P)$
- (9.39) Provided $\neg\text{occurs}('x', 'P')$, $(\exists x \mid : P) \equiv P$
- (9.40) **Interchange of quantifications:**
Provided $\neg\text{occurs}('y', 'R')$ and $\neg\text{occurs}('x', 'Q')$,
 $(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$



- (9.41) **Metatheorem Witness:** Provided $\neg occurs('x', 'P, Q, R')$,
 $(\exists x \mid R : P) \Rightarrow Q$ is a theorem iff $(R \wedge P) \Rightarrow Q$ is a theorem.

SET THEORY

SET COMPREHENSION AND ENUMERATION

- (10.2) **Set enumeration:**
 $\{E_0, \dots, E_{n-1}\} = \{x \mid x = E_0 \vee \dots \vee x = E_{n-1} : x\}$
- (10.3) **Axiom, Universe:** $\mathbf{U} : set(t) = \{x : t \mid true : x\}$
- (10.4) **Axiom, Set membership:** Provided $\neg occurs('x', 'F')$,
 $F \in \{x \mid R : E\} \equiv (\exists x \mid R : F = E)$
- (10.5) **Axiom, Extensionality:** $S = T \equiv (\forall x \mid : x \in S \equiv x \in T)$
- (10.6) $S = \{x \mid x \in S : x\}$
- (10.7) **Empty range:** $\{x \mid false : P\} = \{\}$
- (10.8) **Empty set:** $(v \in \{x \mid false : P\}) = (v \in \{\}) = false$
- (10.9) **Singleton membership :** $e \in \{E\} \equiv e = E$
- (10.10) **One-point rule:** Provided $\neg occurs('x', 'E')$,
 $\{x \mid x = E : P\} = \{P[x:=E]\}$
- (10.11) **Dummy reordering:** $\{x, y \mid R : P\} = \{y, x \mid R : P\}$
- (10.12) **Change of dummy:** Provided $\neg occurs('y', 'R, P')$
and f has an inverse,
 $\{x \mid R : P\} = \{y \mid R[x:=f.y] : P[x:=f.y]\}$
- (10.13) **Dummy renaming:** Provided $\neg occurs('y', 'R, P')$,
 $\{x \mid R : P\} = \{y \mid R[x:=y] : P[x:=y]\}$
- (10.14) Provided $\neg occurs('y', 'R, E')$, $\{x \mid R : E\} = \{y \mid (\exists x \mid R : y = E)\}$
- (10.15) $x \in \{x \mid R\} \equiv R$
- (10.16) **Principle of comprehension.** To each predicate R corresponds the set $\{x : t \mid R\}$, which contains the objects in t that satisfy characteristic predicate R of the set.
- (10.17) $\{x \mid Q\} = \{x \mid R\} \equiv (\forall x \mid : Q \equiv R)$
- (10.18) **Metatheorem:** $\{x \mid Q\} = \{x \mid R\}$ is valid iff $Q \equiv R$ is valid.

OPERATIONS ON SETS

- (10.20) **Axiom, Size:** $\#S = (\Sigma x \mid x \in S : 1)$
- (10.21) **Axiom, Subset:** $S \subseteq T \equiv (\forall x \mid x \in S : x \in T)$
- (10.22) **Axiom, Proper subset:** $S \subset T \equiv S \subseteq T \wedge S \neq T$
- (10.23) **Axiom, Superset:** $T \supseteq S \equiv S \subseteq T$
- (10.24) **Axiom, Proper superset:** $T \supset S \equiv S \subset T$
- (10.25) **Axiom, Complement:** $x \in \sim S \equiv x \notin S$ (for x in \mathbf{U})
- (10.26) **Axiom, Union:** $v \in S \cup T \equiv v \in S \vee v \in T$



(10.27) **Axiom, Intersection:** $v \in S \cap T \equiv v \in S \wedge v \in T$

(10.28) **Axiom, Difference:** $v \in S - T \equiv v \in S \wedge v \notin T$

(10.29) **Axiom, Power set:** $v \in \mathcal{P}S \equiv v \subseteq S$

(10.31) **Metatheorem:** For any set expressions E_s and F_s ,

(a) $E_s = F_s$ is valid iff $E_p \equiv F_p$ is valid,

(b) $E_s \subseteq F_s$ is valid iff $E_p \Rightarrow F_p$ is valid,

(c) $E_s = \mathbf{U}$ is valid iff E_p is valid.

PROPERTIES OF COMPLEMENT

(10.32) $\sim\{x:t \mid P\} = \{x:t \mid \neg P\}$

(10.33) $\sim\sim S = S$

(10.34) $\sim\mathbf{U} = \{\}$

(10.35) $\sim\{\} = \mathbf{U}$

PROPERTIES OF UNION AND INTERSECTION

(10.36)–(10.40): \cup is symmetric, associative, idempotent, has zero \mathbf{U} ,
and has identity $\{\}$.

(10.41) **Weakening:** $S \subseteq S \cup T$

(10.42) **Excluded middle:** $S \cup \sim S = \mathbf{U}$

(10.43)–(10.47): \cap is symmetric, associative, idempotent, has zero $\{\}$,
and has identity \mathbf{U} .

(10.48) **Strengthening:** $S \cap T \subseteq S$

(10.49) **Contradiction:** $S \cap \sim S = \{\}$

(10.50) **Distributivity of \cup over \cap :** $S \cup (T \cap V) = (S \cup T) \cap (S \cup V)$

(10.51) **Distributivity of \cap over \cup :** $S \cap (T \cup V) = (S \cap T) \cup (S \cap V)$

(10.52) **De Morgan:** (a) $\sim(S \cup T) = \sim S \cap \sim T$

(b) $\sim(S \cap T) = \sim S \cup \sim T$

(10.53) $S \subseteq T \wedge W \subseteq V \Rightarrow (S \cup W) \subseteq (T \cup V)$

(10.54) $S \subseteq T \wedge W \subseteq V \Rightarrow (S \cap W) \subseteq (T \cap V)$

(10.55) $S \subseteq T \equiv S \cup T = T$

(10.56) $S \subseteq T \equiv S \cap T = S$

(10.57) $S \cup T = \mathbf{U} \equiv (\forall x \mid x \in \mathbf{U} : x \notin S \Rightarrow x \in T)$

(10.58) $S \cap T = \{\} \equiv (\forall x \mid x \in S \Rightarrow x \notin T)$

(10.59) **Range split:** $\{x \mid P \vee Q : E\} = \{x \mid P : E\} \cup \{x \mid Q : E\}$

PROPERTIES OF SET DIFFERENCE

(10.60) $S - T = S \cap \sim T$

(10.61) $S - T \subseteq S$

(10.62) $S - \{\} = S$



- (10.63) $S - \mathbf{U} = \emptyset$
(10.64) $\{\} - S = \{\}$
(10.65) $\mathbf{U} - S = \sim S$
(10.66) $S \cap (T - S) = \{\}$
(10.67) $S \cup (T - S) = S \cup T$
(10.68) $S - (T \cup V) = (S - T) \cap (S - V)$
(10.69) $S - (T \cap V) = (S - T) \cup (S - V)$
(10.70) $(S \cup T) - V = (S - V) \cup (T - V)$
(10.71) $(S \cap T) - V = (S - V) \cap (T - V)$
(10.72) $\sim(S - T) = \sim S \cup T$

PROPERTIES OF SUBSET

- (10.73) $(\forall x \mid P \Rightarrow Q) \equiv \{x \mid P\} \subseteq \{x \mid Q\}$
(10.74) **Antisymmetry:** $S \subseteq T \wedge T \subseteq S \equiv S = T$
(10.75) **Reflexivity:** $S \subseteq S$
(10.76) **Transitivity:** $S \subseteq T \wedge T \subseteq V \Rightarrow S \subseteq V$
(10.77) $\{\} \subseteq S$
(10.78) $S \subset T \equiv S \subseteq T \wedge \neg(T \subseteq S)$
(10.79) $S \subset T \equiv S \subseteq T \wedge (\exists x \mid x \in T : x \notin S)$
(10.80) $S \subseteq T \equiv S \subset T \vee S = T$
(10.81) $S \not\subseteq S$
(10.82) $S \subset T \Rightarrow S \subseteq T$
(10.83) $S \subset T \Rightarrow T \not\subseteq S$
(10.84) $S \subseteq T \Rightarrow T \not\subseteq S$
(10.85) $S \subseteq T \wedge \neg(V \subseteq T) \Rightarrow \neg(V \subseteq S)$
(10.86) $T \subseteq U \wedge \neg(T \subseteq S) \Rightarrow \neg(U \subseteq S)$
(10.87) **Transitivity:** (a) $S \subseteq T \wedge T \subseteq V \Rightarrow S \subseteq V$
(b) $S \subset T \wedge T \subseteq V \Rightarrow S \subset V$
(c) $S \subset T \wedge T \subset V \Rightarrow S \subset V$

PROPERTIES OF POWERSET

- (10.88) $\mathcal{P}\{\} = \{\{\}\}$
(10.89) $S \in \mathcal{P}S$
(10.90) $\#(\mathcal{P}S) = 2^{\#S}$ (for finite set S)
(10.91) **Monotonic \mathcal{P} :** $S \subseteq T \Rightarrow \mathcal{P}S \subseteq \mathcal{P}T$
(10.95) **Axiom of Choice:** For t a type, there exists a function $f: \text{set}(t) \rightarrow t$ such that for any nonempty set S , $f.S \in S$.



MATHEMATICAL INDUCTION

- (11.3) **Mathematical Induction over \mathbb{N} :** For all P ,
 $P.0 \wedge (\forall n:\mathbb{N} \mid (\forall i \mid 0 \leq i \leq n : P.i) \Rightarrow P(n+1)) \Rightarrow (\forall n:\mathbb{N} \mid P.n)$
- (11.7) **Heuristic:** In proving $(\forall i \mid 0 \leq i \leq n : P.i) \Rightarrow P(n+1)$
by assuming $(\forall i \mid 0 \leq i \leq n : P.i)$, manipulate or restate
 $P(n+1)$ in order to expose at least one of $P.0, \dots, P.n$.
- (11.16) **Mathematical Induction over $\langle U, \prec \rangle$:** For all P ,
 $(\forall x \mid P.x) \equiv (\forall x \mid (\forall y \mid y \prec x : P.y) \Rightarrow P.x)$
- (11.18) **Definition of well-founded:** $\langle U, \prec \rangle$ is well-founded iff, for all S ,
 $S \neq \{\}$ $\equiv (\exists x \mid x \in S \wedge (\forall y \mid y \prec x : y \notin S))$
- (11.19) **Theorem:** $\langle U, \prec \rangle$ is well founded iff it admits induction.
- (11.21) **Axiom, Finite chain property:**
 $(\forall x \mid (\forall y \mid y \prec x : DCF.y) \Rightarrow DCF.x)$
- (11.23) **Theorem:** $\langle U, \prec \rangle$ is well founded iff every decreasing chain is finite.