This is a 50-minute in class closed book exam. All questions are straightforward and you should have no trouble doing them. Please show all work and write legibly. Thank you.

1. (a) Give a context-free grammar for the language
   \[ \{ x \mid x \in (a + b)^* \text{ and } x \text{ has an equal number of } a's \text{ and } b's \} \]

(b) For each variable in the grammar describe the strings that can be generated from the variable by a statement such as

   \[ S \Rightarrow x \text{ iff } x \text{ has an equal number of } a's \text{ and } b's \]

**Solution:**

\[ S \rightarrow \varepsilon | SS | aSb | bSa \]

\[ S \Rightarrow x \text{ iff } x \text{ has an equal number of } a's \text{ and } b's \]

Clearly any string generated has an equal number of a’s and b’s since the righthand side of each production has an equal number of a’s and b’s. Thus, it remains to show only that all such strings are generated.

Let x have an equal number of a’s and b’s. If \( x = \varepsilon \) then clearly x is generated. Assume all x with equal number of a’s and b’s and \( |x| < n \) are generated. Let x be of length n. Write \( x = x_1x_2 \cdots x_k \) where each \( x_i \) has an equal number of a’s and b’s. If \( k \geq 2 \) then each \( |x_i| < n \) and hence can be generated by \( S \Rightarrow SS \cdots \Rightarrow x_1x_2 \cdots x_k \). If no proper prefix of x has an equal number of a’s and b’s then x either starts with an a and ends with a b or vice versa. In this case \( S \Rightarrow aSb \Rightarrow x \) or \( S \Rightarrow bSa \Rightarrow x \).

**Alternative solutions:**

\[ S \rightarrow a | b | SS | aB | bA \]

\[ A \rightarrow a | bAA \]

\[ B \rightarrow b | aBB \]

\[ S \Rightarrow x \text{ iff } x \text{ has an equal number of } a's \text{ and } b's \]

\[ A \Rightarrow x \text{ iff } x \text{ has one more } a \text{ than } b \text{ and no proper prefix of } x \text{ has this property } \]

\[ B \Rightarrow x \text{ iff } x \text{ has one more } b \text{ than } a \text{ and no proper prefix of } x \text{ has this property } \]

\[ S \rightarrow aSbS | bSaS | \varepsilon \]

2. Use the pumping lemma to prove that the language \( \{ ww \mid w \in (a + b)^* \} \) is not a context-free language.
Solution: Let \( n \) be the constant of the pumping lemma. Select \( z = a^n b^n a^n b^n = uvwxvy, \quad vx \neq \epsilon \quad |vx| \leq n \). If \( vx \in a^+ \), \( uv^2wxy \) is either \( a^{n+k} b^n a^n b^n \) or \( a^n b^n a^{n+k} b^n \). Since \(|s| > n\), \( s \) must end in \( b^n \) and only one copy of \( s \) has \( n \) a’s and the other \( n+k \) a’s.

If \( vx \in b^+ \), \( uv^2wxy \) is either \( a^n b^n a^n b^n \) or \( a^n b^n a^n b^{n+k} \). Since \(|s| > n\), \( s \) must start with \( a^n \) and only one copy of \( s \) has \( n \) b’s and the other \( n+k \) b’s.

If \( vx \) in \( a^+ b^+ \), \( uwy \) is either \( a^{n-k} b^{n-k} a^n b^n \) or \( a^n b^n a^{n-k} b^{n-l} \). Since \(|s| > n\) in the first case \( s \) must end in \( n \) b’s and there are not enough b’s for the first \( s \). In the second case \( s \) must start with \( n \) a’s and there are not enough a’s for the second \( s \).

If \( vx \) is in \( b^+ a^+ \) then \( uwy = a^n b^{n-k} a^{n-l} b^n \). Again \( s \) must end in \( n \) b’s and there are not enough b’s for the first \( s \).

3. (a) What is the meaning of a symbol of the form \([qAp]\) in the conversion of a pda to a cfg?

**Solution:** A string \( x \) can be derived from the symbol \([qAp]\) iff the pda starting in state \( q \) with \( A \) on top of the pushdown store processes the input \( x \) ultimately exposing the symbol below \( A \) on the store in state \( p \).

(b) Suppose \( \delta(q, a, A) \) contains \((s, A_1 A_2 \cdots A_k)\). What productions does this give rise to in the grammar?

**Solution:** For each \( p, r_1, r_2, \cdots, r_{k-1} \) in \( Q \) create production

\[ [qAp] \rightarrow a[rA_1 r_1][r_1 A_2 r_2] \cdots [r_{k-1} A_k p] \]

(c) Suppose \( \delta(q, a, A) \) contains \((s, \epsilon)\). What productions does this give rise to in the grammar?

**Solution:** \([qAr] \rightarrow a\)