**Problem 2** Suppose the input is $w$, $|w| = n$. The intuition is that we have to guess the function $\delta^n$ i.e. what states can be reached from any state on an input of length $n$. Suppose $Q = \{q_1, \ldots, q_k\}$ is the set of states of the finite automata accepting the original regular language. Let each state of the new finite automata be of the form

$$(q, q'', Q_1, \ldots, Q_k, R_1, \ldots, R_k), \quad q \in Q, Q_i \subseteq Q, R_i \subseteq Q.$$  

The start state is $q = q_{\text{start}}$, $q'' = q_{\text{final}}$, and each $Q_i = R_i = \emptyset$. From there, we take an $\varepsilon$-transition where $Q_i$ is guessed to be a subset of $Q$, but all the $R_i = \{i\}$ and $q$ and $q''$ values remain same i.e. $q_{\text{start}}$ and $q_{\text{final}}$ respectively. The intuition is that we start out by guessing the mapping $Q$ which captures where each state $i$ can possibly go on a string of length $|w|$.

Now, on each input, the transition function is defined as

$$\delta((q, q', Q_1, \ldots, Q_k, R_1, \ldots, R_k), a) \in \{\delta(q, a), q'', Q_1, \ldots, R'_1, \ldots, R'_k) \mid R'_i = \{\delta(R_i, b), b \in \Sigma\}, q'' \in \{q_i \in Q_i \mid q' \} \}$$

The final state is any state such that the following conditions hold : i) $q = q'$. ii) For each $i$, $R_i = Q_i$. We go forward using the state $q$. We take steps of length $|x|$ each in reverse using the $Q_i$ mapping. Finally, the string is accepted only if there is the original mapping has been guessed correctly, and the state that prefix $w$ reaches is also matched by the reversed path of the automata.

**Theorem 0.1** The language accepted by this automata is exactly the language $R = \{w \mid \exists x, |x| = |w|^2, wx \in L\}$.

**Proof.** We do this by induction. The induction statement is however not about acceptance. All we claim as induction statement, is that the after reading a string of length $|w|$, each set $R_i$ satisfies the condition that there $R_i = \{r : \exists y, |y| = |w|, r = \delta(i, y)\}$. It should be clear that once we prove this claim, the rest follows easily.

As always, for the base case let $w = \varepsilon$. Each $R_i = \{i\}$ and continues to be so, since we start with a DFA for $L$. The hypothesis is thus trivially true.

Suppose now, that the string $w$ is accepted by our NFA. Then, after reading $w$, we have a state $(q, q', Q_1, \ldots, R_i, \ldots)$ that satisfies the two conditions, : i) $q = q'$. ii) For each $i$, $R_i = Q_i$. Since, each $R_i = \{r : \exists y, |y| = |w|, r = \delta(i, y)\}$, so is each $Q_i$. But, the state
q' (the second element of our state vector) has been reached at by starting at $q_{\text{final}}$ and then taking $|w|$ reverse transitions using the map $Q$. Thus, $\exists z = x_1...x_{|w|}$, each $x_i$ satisfying $|x_i| = |w|$, such that $\hat{\delta}(q', z) = q_{\text{final}}$. Since $q = q'$, and $\hat{\delta}(q_{\text{start}}, w) = q$, we have that $\hat{\delta}(q_{\text{start}}, wz) = \hat{\delta}(q', z) = q_{\text{final}}$, and $|z| = |w|^2$. Hence we only accept strings of this form.

The other direction, that is, if there exists a $w$ that satisfies the given condition, there exists a accepting path for it is easy to prove by just choosing the mapping $Q$ correctly.

$\square$