Problem 1

We are interested in finding a CFG for the complement of $S = \{01001...0^i10^i1 | i \geq 1\}$. To do this, let us characterize the set $S$ in the following way:

1. Strings in $S$ begin with $01$.
2. Strings in $S$ end with $1$.
3. Strings have pieces of the form $0^i10^{i+1}$ with $j = i + 1$.

Then, to get the complement of $S$, we must create a CFG to break each of these properties.

To break property 1, we simply have:

$$S \rightarrow 1A \mid 00A$$
$$A \rightarrow 1A \mid 0A \mid \varepsilon$$

Similarly, for property 2, we have:

$$S \rightarrow A0$$

Now, to break property three, we must make sure that, somewhere in a string generated by the CFG, there is a substring of the form $10^i10^{i+1}$ with $j \neq i + 1$. We will have to break this further into two cases; first, we must consider substrings of the form $10^i10^{i+1}$ with $i \geq j$ and then consider $10^i10^{i+1}$ with $j > i + 1$. This can occur anywhere in the string, so for the first case we have

$$S \rightarrow A1BC1A$$
$$B \rightarrow 0B \mid \varepsilon$$
$$C \rightarrow 0C0 \mid 1$$

Similarly, for the second case we have

$$S \rightarrow A1C00B1A$$

However, there is one problem. These productions assume that there are at least three ones in the string. We must also handle the cases of one and two 1’s. For one 1 we want $0^*10^*-01$ and for two 1’s we want $0^*10^*10^*01001$. We can easily write a CFG to handle these cases, and we work it into our grammar as follows:

$$S \rightarrow E$$
$$E \rightarrow 00B1B$$

For the case of one 1. We require that there are at least two zeros before the one (if it starts with a one, we already handle that case from property 1 above). Similarly, for two 1’s:

$$S \rightarrow F$$
$$F \rightarrow 001B1b \mid B101B \mid B1000B1B$$

Putting this all together gives us the CFG for the complement of $S$. 