7.3.4 d

Define homomorphisms $h_1$, $h_2$, $h_3$:

\[
\begin{align*}
    h_1(\hat{\Sigma}) &= \epsilon \\
    h_1(\Sigma) &= \Sigma \\
    h_2(\hat{\Sigma}) &= \Sigma \\
    h_2(\Sigma) &= \epsilon \\
    h_3(\hat{\Sigma}) &= \Sigma \\
    h_3(\Sigma) &= \Sigma
\end{align*}
\]

Then $h_{1}^{-1}(L) = A$ is a set of strings consisting of symbols with and without hats such that if you consider in any particular string only the symbols without hats, the string consisting of these unhatted letters in the order they appear in the original string is in $L$. Note that $A$ is a CFL by 7.30. Similarly $h_{2}^{-1}(R) = B$ is a set of strings consisting of symbols with and without hats such that if you consider in any particular string only the symbols having hats, the string consisting of these hatted symbols in the order they appear in the original string is in $R$ (without the hats, of course). Note that $B$ is a regular language by 4.16. The intersection $A \cap B = C$ is a set of strings such that for any particular string the concatenation of all hatted symbols relative to the order of the original string is a string in $R$ (without the hats) and the concatenation of all unhatted symbols relative to the order of the original string is a string in $L$. In addition, the hatted symbols are shuffled with the unhatted symbols. Note that $C$ is a CFL by 7.27. In order to obtain $\text{shuffle}(L, R)$ we need to remove all the hats from the strings in $C$. So $h_{3}(C)$ gives us $\text{shuffle}(L, R)$, and so $\text{shuffle}(L, R)$ is a CFL by 7.24.4.
Suppose \( \text{shuffle}(L_1, L_2) \) is a CFL for any CFLs \( L_1 \) and \( L_2 \). Let \( L_1 = \{a^n b^n | n \geq 1 \} \). Let \( L_2 = \{c^n d^n | n \geq 1 \} \). Let \( n \) be the constant of the pumping lemma for \( \text{shuffle}(L_1, L_2) \). Take \( z = a^n c^n b^n d^n \), clearly a member of \( \text{shuffle}(L_1, L_2) \). Then \( z = uvwx y \) such that \( |vwx| \leq n \), \( vx \) is not \( \epsilon \), and for all \( i \), \( uv^i w x^i y \) is in \( \text{shuffle}(L_1, L_2) \). Note that since \( |vwx| \leq n \), \( vw x \) contains at most two distinct symbols. Let \( i = 2 \), then it must be the case that the resulting string has either a different number of a’s compared to the number of b’s or has a different number of c’s compared to the number of d’s (this can be checked by cases), so this resulting string is not in \( \text{shuffle}(L_1, L_2) \), contradicting the pumping lemma. So our assumption was false, and \( \text{shuffle}(L_1, L_2) \) is not a CFL. So in general \( \text{shuffle}(L_1, L_2) \) may not be a CFL for two arbitrary CFLs \( L_1 \) and \( L_2 \).