7.2.1 e, f

Use the CFL pumping lemma to show that each of the following languages is not context-free:

e) \{a^n b^i c^j \mid n \leq i \leq 2n \}

1. Pick \( L = \{a^n b^i c^j \mid n \leq i \leq 2n \} \)
2. The demon (our opponent) gets to pick \( n \)
3. We pick \( z = a^n b^i c^j \)
4. The demon gets to break \( z \) into \( uvwx \) s.t. \(|vwx| \leq n\) and \( vx \neq \epsilon \)
5. We “win” the game by picking \( i \) and showing that \( uv^i wx^i y \) is not in \( L \)

\( vwx \in a^+ \)
   Pick \( i = 2 \). Then \( z = a^{n+|vx|} b^n c^n \notin L \)

\( vwx \in b^+ \)
   Pick \( i = 2 \). Then \( z = a^n b^{n+|vx|} c^n \notin L \)

\( vwx \in c^+ \)
   Pick \( i = 0 \). Then \( z = a^n b^n c^{n-|vx|} \notin L \)

\( vwx \in a^+ b^+ \)
   Pick \( i = 2 \). Then \( z = a^p b^q c^n \), where \( p + q = 2n + |vx| \). At least one \( a \) was added so \( p > n \), which means there are more \( a \)'s than \( c \)'s, so \( z \notin L \)

\( vwx \in b^+ c^+ \)
   Pick \( i = 0 \). Then \( z = a^n b^p c^q \), where \( p + q = 2n - |vx| \). At least one \( b \) was removed so \( p < n \), which means there are more \( a \)'s than \( b \)'s, so \( z \notin L \)

f) \{wwRw \mid w \text{ is a string of 0's and 1's} \}

1. Pick \( L = \{wwRw \mid w \text{ is a string of 0's and 1's} \} \)
2. The demon gets to pick \( n \)
3. We pick \( w = 0^n 1^n \), so \( z = 0^n 1^n 1^n 0^n 0^n 1^n = 0^n 1^{2n} 0^{2n} 1^{2n} \)
4. The demon gets to break \( z \) into \( uvwx \) s.t. \(|vwx| \leq n\) and \( vx \neq \epsilon \)
5. We “win” the game by picking \( i \) and showing that \( uv^i wx^i y \) is not in \( L \)

\( vwx \in 0^+ \)
   Pick \( i = 0 \). Then either \( z = 0^{n-|vx|} 1^{2n} 0^{2n} 1^n \) or \( z = 0^n 1^{2n} 0^{2n-|vx|} 1^n \). In either case the number of 0’s in \( z \) is strictly less than the number of 1’s, so \( z \notin L \).
$\forall vwx \in 1^+$

Pick $i = 0$. Then either $z = 0^n 1^{2n-|vx|} 0^{2n} 1^n$ or $z = 0^n 1^{2n} 0^{2n} 1^{n-|vx|}$. In either case the number of 0’s in $z$ is strictly greater than the number of 1’s, so $z \not\in L$.

$\forall vwx \in 0^+1^+$

Pick $i = 0$. Then either $z = 0^p 1^n q 0^{2n} 1^n$ or $z = 0^{2n} 1^n 0^p 1^{n+q}$, where $p + q = 2n - |vx|$. In the case of $z = 0^p 1^n q 0^{2n} 1^n$, the first third of $z$ contains at least $n + 1$ 0’s, which means it contains at most $n - 1$ 1’s. Since the number of 1’s is not equal in the first and third 1/3rd of $z$, $z \not\in L$. A similar argument works for $z = 0^{2n} 1^n 0^p 1^{n+q}$.

$\forall vwx \in 1^0^+$

Pick $i = 0$. Then $z = 0^n 1^p 0^q 1^n$, where $p + q = 4n - |vx|$. If $p \neq q$ then clearly $z \not\in L$. If $p = q$ then $z = 0^n 1^{2n - |vx|/2} 0^{2n - |vx|/2} 1^n$. Now the length of $z$ is $6n - |vx|$, so one third of $z$ is $2n - |vx|/3$. If we split $z$ into three equal length parts $w_1$, $w_2$, and $w_3$, where $z = w_1 w_2 w_3$ then $w_1 = 0^n 1^{n - |vx|/3}$ and $w_3 = 0^n 1^{n - |vx|/3} 1^n$, so $w_2 = 1^n - |vx|/6 0^n - |vx|/6$. $w_1 w_2 w_3 \not\in L$.