Problem 4

Write a CFG $G$ for: $\{ x > y^R \mid x$ is an ID of a given Turing machine $M$, $>$ is a special symbol that separates ID’s, and $y$ is the successor ID to $x}\$

Let $M = \{Q, \Sigma, \Gamma, \delta, q_0, B, F\}$

We can place two restrictions on $M$:
1. $> \notin \Gamma$ the special symbol $>$ does not appear in either ID
2. $Q \cap \Gamma = \emptyset$ “we do not use as a state any symbol that is also a tape symbol”

A bit of notation:
From now on we will let $X, Y, X_i, \text{etc.}$ stand for “any symbol in $\Gamma$”. We will use $X$ in regular expressions and grammar productions with the knowledge that we could easily replace $X$ with the disjunction of all of the symbols in $\Gamma$. The same thing applies to $p, q \in Q$

Let’s write a regular expression for an ID as a little warm-up:
An ID is a finite string of tape symbols followed by a state followed by another finite string of tape symbols so we have: $X^*pY^*$

Now we are ready to construct the CFG $G$:

Let $G = \{V, T, P, S\}$ where

$T = \Gamma \cup Q \cup \{>\}$ The terminal symbols are those that can appear in an ID ($\Gamma \cup Q$), along with the special symbol $>$. Note that safely forming $T$ is where the two restrictions from above come into play.

$V = \{S, R, C, D\}$ (see the productions below)

$P$ is constructed as follows:

$S \rightarrow C \mid D$ $D$ is used by some of the special cases below
$C \rightarrow X C X$ $C$ matches the outer (leftmost and rightmost) symbols
$R \rightarrow X R X \mid >$ $R$ matches the innermost symbols (to the left and right of $>$)

For each transition $\delta(q, X_i) = (p, Y, L)$ add:

$C \rightarrow X_{i-1} q X_i R Y X_{i-1} p$

Special case $i = 1$, $\delta(q, X_1) = (p, Y, L)$ “swallow a blank”
$D \rightarrow q X_1 R Y B p$ (note that we start at $D$ because there should be nothing before $q$)

Special case $i = n$ and $Y = B$, $\delta(q, X_n) = (p, B, L)$ append $B$ to the infinite sequence of trailing blanks

$C \rightarrow X_{n-1} q X_n > X_{n-1} p$
For each transition $\delta(q, X_i) = (p, Y, R)$ add:

$C \rightarrow q \ X_i \ R \ p \ Y$

Special case $i = n$, $\delta(q, X_i) = (p, Y, R)$ head ends up at a $B$

$C \rightarrow q \ X_n \ B \ p \ Y$

Special case $i = 1$ and $Y = B$, $\delta(q, X_i) = (p, B, R)$ append $B$ to the infinite sequence of leading blanks

$D \rightarrow q \ X_i \ R \ p$ (note that we start at $D$ because there should be nothing before $q$)

Example 1:

Suppose $\delta(q, X_i) = (p, Y, L)$, then

$X_1 \ X_2 \ldots X_{i-2} \ q \ X_i \ X_{i+1} \ldots X_n >_M X_1 \ X_2 \ldots X_{i-2} \ q \ X_i \ X_{i+1} \ldots X_n$

we want to accept:

$q \ X_1 \ X_2 \ldots X_n \ X_{i+1} \ldots X_1 \ Y \ X_{i+1} \ldots X_n \ q \ X_{i+2} \ldots X_2 \ X_1$

so start with $S \rightarrow C$, then

$C \rightarrow X \ C \ X$ matches $X_1 \ X_2 \ldots X_{i-2} \ C \ X_{i-2} \ldots X_2 \ X_1$

$C \rightarrow X_{i-1} \ q \ X_i \ R \ Y \ X_{i-1} \ p$ matches $X_1 \ X_2 \ldots X_{i-2} \ X_{i-1} \ q \ X_i \ R \ Y \ X_{i-1} \ p \ X_{i-2} \ldots X_2 \ X_1$

$R \rightarrow X \ R \ X$ matches $X_1 \ X_2 \ldots X_{i-2} \ X_{i-1} \ q \ X_i \ X_{i+1} \ldots X_n \ R \ X_n \ldots X_{i-1} \ Y \ X_{i+1} \ p \ X_{i-2} \ldots X_2 \ X_1$

$R \rightarrow >$ matches $X_1 \ X_2 \ldots X_{i-2} \ X_{i-1} \ q \ X_i \ X_{i+1} \ldots X_n >_M X_n \ldots X_{i-1} \ Y \ X_{i-1} \ p \ X_{i-2} \ldots X_2 \ X_1$

Example 2:

Suppose $\delta(q, X_1) = (p, B, R)$, then

$p \ X_1 \ X_2 \ldots X_n >_M p \ X_2 \ldots X_n$

we want to accept:

$p \ X_1 \ X_2 \ldots X_n >_M X_n \ldots X_2 \ p$

so start with $S \rightarrow D$, then

$D \rightarrow q \ X_i \ R \ p$ matches $q \ X_i \ R \ p$

$R \rightarrow X \ R \ X$ matches $q \ X_1 \ X_2 \ldots X_n \ R \ X_n \ldots X_2 \ p$

$R \rightarrow >$ matches $q \ X_1 \ X_2 \ldots X_n >_M X_n \ldots X_2 \ p$