Solutions for 9.2.2bc

For part b, a solid argument would include an inductive proof and that is what I will present here.

Claim I: $\text{Ackermann}(x, 1) = 2x$

**Proof:**

Base Case:

$x = 0 \rightarrow \text{Ackermann}(0, 1) = 0 = 2*0$ by definition.

Inductive Case:

IH: True for all $x < n$.

Claim: True for $x = n$.

\[
\text{Ackermann}(n, 1) = \text{Ackermann}(\text{Ackermann}(n - 1), 0) \quad \text{(by def)}
\]

\[
= \text{Ackermann}(2*(n - 1), 0) \quad \text{(by IH)}
\]

\[
= 2*(n - 1) + 2 \quad \text{(by def)}
\]

\[
= 2*n \quad \text{(QED)}
\]

Theorem: $\text{Ackermann}(x, 2) = 2^x$ (guess after a few experiments)

Base Case:

$x = 0 \rightarrow \text{Ackermann}(0, 2) = 1 = 2^0$ (by def)

Inductive Case:

IH: True for all $x < n$.

Claim: True for $x = n$.

\[
\text{Ackermann}(n, 2) = \text{Ackermann}(\text{Ackermann}(n - 1, 2), 1) \quad \text{(by def)}
\]

\[
= \text{Ackermann}(2^{n-1}, 1) \quad \text{(by IH)}
\]

\[
= 2 * 2^{n-1} \quad \text{(by Claim I)}
\]

\[
= 2^n \quad \text{(QED)}
\]

Since the question did not explicitly ask for a proof, I did not take off points for this. Yet, I gave 2 points of extra credit to those who did the proof. Some of you did not do the proof but gave numerous evaluations and a guess based on the pattern emerged (Proof by many examples 😊). I gave 1 point extra to those. The rest wrote a program in some language to do the experiment. This is still okay but it misses the point of the question completely, which is to see if you understand the evaluation of recursively defined functions. So, no extra credit for those.

Part c was a straightforward evaluation. The fact we just proved makes it a lot easier. Some of you tried to evaluate the full function, which of course is quite tedious. It was okay to use a program to do this part if you have shown part (a) either by proof or by establishing the pattern.

\[
\text{Ackermann}(4, 3) = 2^{2^4} = 2^{16} = 65,536
\]

The full evaluation is (evaluated function instance is highlighted):

\[
A(4,3)
A(A(3,3), 2)
A(A(A(2,3), 2), 2)
A(A(A(A(1,3), 2), 2), 2)
A(A(A(A(A(0, 3), 2), 2), 2), 2) \quad \text{(reached base case, use part a to collapse the rest)}
A(A(A(A(A(1, 2), 2), 2), 2)
A(A(A(2, 2), 2), 2)
A(A(4, 2), 2)
A(16, 2)
2^{16}
\]