Problem 4  This problem did not ask for a proof, so we did not take off points for incorrect proofs, informal proofs, or total lack of a proof.

Many of you argued informally to show your example satisfied (b), with the idea that, “the intersection of a finite number n of sets in W looks like such and such, so as n goes to infinity it must be empty.” Please see the proofs below to see how to prove this more formally.

Solution 1: Let $W = \{L_i | i \in \mathbb{N}\}$, where $L_i = \{w \in \{0,1\}^* | |w| > i\}$. Note: with W of this form, we can naturally associate subsets of W with subsets of $\mathbb{N}$; $V \subseteq W$ corresponds to $S_V = \{i | L_i \in V\}$. And conversely $S \subseteq \mathbb{N}$ corresponds to $\{L_i | i \in S\}$.

Proof that W satisfies (a): Let V be a finite subset of W, and let $S_V$ be the corresponding finite subset of $\mathbb{N}$. Then $S_V$ contains a maximal element, say n. Then, $1^{n+1} \in L_i$ for all $i \in S_V$. Therefore, $1^{n+1} \in \bigcap_{i \in S_V} L_i = \bigcap_{L \in V} L$, so this intersection is non-empty.

Proof that W satisfies (b) and (c): Let T be any subset of $\mathbb{N}$. Assume there exists some string $x \in \bigcap_{i \in T} L_i$. By definition, this means that $x \in L_i$ for all $i \in T$. Suppose $|x| = k$. Then for $i > k$, $i \notin T$. Thus, T must be finite. It follows that any subset of W with nonempty intersection is finite, so all infinite subsets of W have empty intersection. We have now shown (c). For (b), it suffices to note that there exists an infinite subset of W (in particular, W itself is infinite), and by (c), this subset has empty intersection.

Solution 2: Let $W = \{L_i | i \in \mathbb{N}\}$, where $L_i = ((01)^i)^* - \{\varepsilon\}$.

Proof that W satisfies (a): Let S be a finite subset of $\mathbb{N}$, and let $k = \text{LCM}\{i | i \in S\}$. Then $\bigcap_{i \in S} L_i = L_k \neq \emptyset$. Note: It is not the case that $\bigcap_{i \in S} L_i = L_k$, where k is the greatest number in S (this was a common error).

Proof that W satisfies (b) and (c): Let T be a subset of $\mathbb{N}$. Assume there exists some string $x \in \bigcap_{i \in T} L_i$. By definition, this means that $x \in L_i$ for all $i \in T$. Suppose $|x| = k$. Then for $i > k$, $x \notin L_i$, so T must be finite.

Solution 3:

This solution satisfies only (a) and (b), and is somewhat complicated. However, we point out this solution because several students attempted a solution along these lines.
The idea is to let \( L_i \) be the set of all strings in \( \{0,1\}^* \) except \( x_i \), where \( x_i \) is some string. There were several students who had this idea but didn’t explicitly say what \( x_i \) should be. Choosing \( x_i \) appropriately is essential for satisfying (b); for each string \( x \in \{0,1\}^* \) there must be some \( i \) such that \( x_i = x \). In other words, the function \( f \) from \( \mathbb{N} \) to \( \{0,1\}^* \) such that \( f(i) = x_i \) must be onto.

We first define an order on \( \{0,1\}^* \) as follows: define \( x \prec y \iff |x| < |y| \) or \( (|x| = |y| \text{ and the first place from the left at which } x \text{ and } y \text{ differ has a } 0 \text{ in } x \text{ and a } 1 \text{ in } y) \). Observe that this is a well-ordering; in particular, for every string \( x \), there is a string \( y \) such that \( x \prec y \) and there is no \( z \) such that \( x \prec z \prec y \). When this relationship holds, call \( y \) the successor of \( x \).

Next, we define a map \( f \) from \( \mathbb{N} \) to \( \{0,1\}^* \) inductively. First, let \( f(0) = \varepsilon \). Now assume \( f(n) \) is defined for all \( n < m \). Then let \( f(m) \) be the successor of \( f(m-1) \) in \( \{0,1\}^* \), with successor as defined above.

Note that \( f \) is onto.

Let \( x_i = f(i) \). Now, let \( W = \{ L_i \mid i \in \mathbb{N} \} \), where \( L_i = \{0,1\}^* - \{x_i\} \).

We omit the proof that this solution is valid, but (b) holds because \( f \) is onto, so that the intersection of all the languages in \( W \) is empty.

Some solutions submitted along these lines did not define \( f \) explicitly and basically said, “let \( f \) be any one-to-one map from \( \mathbb{N} \) to \( \{0,1\}^* \).” The problem with this is that not all such maps are onto. For example, if \( f(n) = 0^n \), then the intersection of all languages in \( W \) contains any string of all 1’s. Similarly, for any such \( f \) that is not onto, the intersection of all languages in \( W \) is non-empty (it contains any string \( x \) not in the range of \( f \)), and (b) is not satisfied.