**FINAL VERSION**

1. For each of the following languages, find out if it is recursive or not. If it is, describe a total Turing machine that computes it. If not, explain why the existence of such a machine entails a contradiction.

   (a) \( \{0^{n_0}1^{n_1}0^{n_2} \ldots 1^{n_2k} : \text{for all } 0 \leq i \leq 2k, \; n_i \in \mathbb{N} \text{ and } n_0 \text{ is a solution to the equation} \; n_1 \cdot x^{n_2} + n_3 x^{n_4} + n_{2k-1} x^{n_{2k}} = 0 \} \)

   (b) \( \{ M : \epsilon \in L(M) \} \)

   (c) \( \{ M : |L(M)| < 100 \} \) (that is, \( M \) accepts less than 100 strings)

   (d) \( \{(M_1, M_2) : L(M_1) = L(M_2) \} \) (where \( M_1, M_2 \) are Turing machines)

2. (a) Prove that if one changes the definition of Turing machines to allow an infinite set of states \( Q \), then for every \( L \subseteq \{0\}^* \) there exists a total machine that computes it.

   (b) Prove that there exists a language \( L \subseteq \{0\}^* \) such that for every Turing machine (under the usual definition), \( L(T) \neq L \).

3. Prove that the family of all recursive languages is closed under the * operation. Namely, if \( L \) is recursive then so is \( L^* = \{w_1 \ldots w_n : n \in \mathbb{N} \text{ and for all } i, w_i \in L \} \).

4. Given a Turing machine \( T \), let \( \overline{T} \) denote the machine obtained by switching the ‘r’ and ‘s’ states of \( T \). That is, the transition function of \( \overline{T} \), \( \overline{\delta} \), is obtained by replacing each occurrence of \( t \) in the function \( \delta \) of \( T \) by an \( r \) and vice versa. Prove or refute each of the following claims:

   (a) For every \( T, L(\overline{T}) = \overline{L(T)} \) (where \( \overline{L} \) is the complement of a language \( L \)).

   (b) For every pair of machines \( T_1, T_2 \), if \( L(T_1) = L(T_2) \) then \( L(\overline{T_1}) = L(\overline{T_2}) \).

   (c) For every machine \( T \), if \( L(T) \) is recursive, then so is \( L(T) \).