Example 1. Find a Chomsky Normal Form of CFG $S \rightarrow aXbY, X \rightarrow aX \mid \epsilon, Y \rightarrow bY \mid \epsilon$.
Apply an algorithm from HO19.

Step 1: getting rid of $\epsilon$.
$S \rightarrow aXbY \mid ab \mid aXb, \quad X \rightarrow aX \mid a, \quad Y \rightarrow bY \mid b$.

Step 2. Replacing terminals in nontrivial productions.
$S \rightarrow AXbY \mid aBY \mid AXb, \quad X \rightarrow AX \mid a, \quad Y \rightarrow BY \mid b, \quad A \rightarrow a, \quad B \rightarrow b$.

Step 3. Shortening long productions by introducing extra nonterminals.
$S \rightarrow AU \mid AV \mid AW \mid AB, \quad U \rightarrow XV, \quad V \rightarrow BY, \quad W \rightarrow XB, \quad X \rightarrow AX \mid a, \quad Y \rightarrow BY \mid b, \quad A \rightarrow a, \quad B \rightarrow b$.

Example 2. Find a Greibach Normal Form of CFG $S \rightarrow XbY, X \rightarrow aX \mid a, \quad Y \rightarrow bY \mid a$.
The general algorithm converting CFG into GNF is not practical. Use common sense and some tricks instead. The first steps are easy: get rid of $\epsilon$ and terminals other then the first ones in their productions:
$S \rightarrow XbY, \quad X \rightarrow aX \mid a, \quad Y \rightarrow bY \mid a, \quad B \rightarrow b$.

Trace the leftmost substitutions on nonterminals occurring first in productions until a terminal comes first:
$S \rightarrow XBY, \quad X \rightarrow aX \mid a, \quad Y \rightarrow bY \mid a, \quad B \rightarrow b$.

Replace old productions with fist nonterminals by all possible productions obtained above:
$S \rightarrow aBY \mid aXbY, \quad X \rightarrow aX \mid a, \quad Y \rightarrow bY \mid a, \quad B \rightarrow b$.

Example 3. Find a Greibach Normal Form of PAREN-$\{-\epsilon\}$ (an old example in a new light).
We start with the usual CFG $S \rightarrow [S] \mid SS \mid [\ ].$ Getting rid of non-first terminals gives
$S \rightarrow [SR \mid SS \mid [R, \quad R \rightarrow \ ]$. Tracing the leftmost substitutions in $S \rightarrow SS$:
$S \rightarrow SS \rightarrow [SRS \ldots \rightarrow SS \rightarrow [RS$. Replacing the old production by the results of the above tracing:
$S \rightarrow [R \mid [SR \mid [RS \mid [SRS, \quad R \rightarrow \ ]$. Example 4. Prove that $A = \{a^nb^n \mid n \geq 0\}$ is not a CFL (cf. Kozen, p.154, Ex. 22.3). Suppose $A$ is a CFL. By the Pumping Lemma there should be an integer $k \geq 1$ such that any $z \in A$ can be broken into $z = uvwx, \ |vx| \geq 1, \ |uw| \leq k$ such that $uv^iwx^iy \in A$ for any $i \geq 0$.
Take $n > k$ and consider all possible partitions $uvwxy$ of the string $a^nb^n\epsilon$.

Case 1. Each of $v, x$ is inside some block of letter. Note, that one of the blocks of letters remains $v, x$-free. Then $uv^iwx^iy$ makes the blocks containing $v, x$ larger, then the third block, therefore, $uv^iwx^iy \notin A$.

Case 2. At least one of $v, x$ has intersections with two different blocks. Then either $v$ or $x$ contains both $a$’s and $b$’s and $uv^iwx^iy$ for $i \geq 2$ is not of the form $a^nb^n\epsilon$ and, therefore, is not in $A$.

Example 5. Prove that $B = \{a^nb^n\epsilon \mid n \geq 0\}$ is not a CFL. $A = h(B)$ where $A$ in from Example 4, and $h$ is homomorphism $h(\epsilon) = a$. Since CFLs are closed under homomorphisms, if $B$ were context free, then $A$ should be too, which is not the case.
Example 6. Assume that \( C = \{a^mb^n | m, n \geq 0\} \) is not a CFL (HW9, cf. Kozen, p.154, Example 22.4). Prove that \( D = \{ww \mid w \in \{a, b\}^*\} \) is not a CFL. Use the theorem that intersection of a CFL with a regular language is a CFL. Note, that \( C \) would be also a CFL, if \( D \) were CFL, then \( C \) would be CFL too.

Example 7. Another example on the same lemma that CFL \( \cap \) REG=\( \)CFL. We establish that the set \( E \) of all strings over \( \{a, b, c\} \) containing equal numbers of \( a, b \) and \( c \) is not a CFL. Note that \( B = E \cap a^*b^*c^* \). Since the set \( a^*b^*c^* \) is regular, if \( E \) were CFL, then \( B \) would be also a CFL, which contradicts Example 5.

Example 8. Convert an NPDA into a CFG. Let an NPDA \( M \) have the transition function \( \delta(s, a, \perp) = (s, XX) \), \( \delta(s, a, X) = (s, X) \), \( \delta(s, b, X) = (t, e) \), \( \delta(t, a, X) = (t, c) \).

We first have to convert \( M \) into a one-state NPDA \( M' \). Use the algorithm from HO24. The corresponding \( \delta' \) is

\[
\delta'(s, a, <s \perp s>) = (*, <sXs><sXs>), \quad \delta'(s, b, <s \perp s>) = (*, <sXt><tXs>)
\]

Tracing for \( M \) on the input \( aaba \):

\[
(s, aaba, \perp) \xrightarrow{1} (s, aba, XX) \xrightarrow{1} (s, ba, XX) \xrightarrow{1} (t, a, X) \xrightarrow{1} (t, e, e)
\]

The corresponding tracing for \( M' \) is:

\[
(*, aaba, <s \perp t>) \xrightarrow{1} (*, aba, <sXt><tXt>) \xrightarrow{1} (*, ba, <sXt><tXt>) \xrightarrow{1} (*, a, <tXt>) \xrightarrow{1} (*, e, e)
\]

From \( M' \) we read the grammar \( G \)

\[
<s \perp s> \rightarrow a <sXs><sXs>, \quad <s \perp s> \rightarrow a <sXt><tXs>
\]

\[
<s \perp t> \rightarrow a <sXs><sXt>, \quad <s \perp t> \rightarrow a <sXt><tXt>,
\]

\[
<sXs> \rightarrow a <sXs>, \quad <sXt> \rightarrow a <sXt>,
\]

\[
<sXt> \rightarrow b, \quad <tXt> \rightarrow a.
\]

Here is how \( G \) generates the string \( aaba \) from the initial nonterminal \( <s \perp t> \):

\[
<s \perp t> \xrightarrow{1} a <sXt><tXt> \xrightarrow{1} aa <sXt><tXt> \xrightarrow{1} aab <tXt> \xrightarrow{1} aaba.
\]

Note that \( M' \) contains many redundancies. After pruning transitions that never apply in accepting computations, we get much shorter NPDA:

\[
\delta'(s, a, <s \perp t>) = (*, <sXt><tXt>), \quad \delta'(s, a, <sXt>) = (*, <sXt>),
\]

\[
\delta'(s, b, <sXt>) = (*, e), \quad \delta'(s, a, <tXt>) = (*, e).
\]

The same holds for the grammar. Here is a short equivalent of \( G \):

\[
<s \perp t> \rightarrow a <sXt><tXt>, \quad <sXt> \rightarrow a <sXt>, \quad <sXt> \rightarrow b, \quad <tXt> \rightarrow a.
\]

Replace awkward notation of nonterminals by the usual upper case Latin letters:

\[
S \rightarrow aVW, \quad V \rightarrow aV | b, \quad W \rightarrow a.
\]

Abandon Greibach Normal Forms and get a shorter CFG:

\[
S \rightarrow aVa, \quad V \rightarrow aV | b.
\]

A direct analysis of derivations allows us to come with yet shorter version:

\[
S \rightarrow aS | aba.
\]